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ABSTRACT

This is the instructor's guide to a text designed for the contemporary business student who frequently questions the need for studying mathematics. The text material is presented in a "building-block" fashion, with skills learned in earlier chapters organized to provide a smooth transition into more difficult topics. This guide consists of sections which parallel the text organization. Each chapter opens with a short paragraph describing the nature and rationale of the corresponding portion of the student text. The bulk of each section in the instructor's guide deals with solutions and problem-solving methods for the exercises found in the student text. Topics covered are: (1) Review of Algebra, (2) Systems of Linear Equations, (3) Matrix Algebra, (4) Matrix Inversion and Applications, (5) Linear Programming-Graphing Concept, (6) Linear Programming, (7) Nonlinear Polynomial Functions, (8) Exponential and Logarithmic Functions, (9) Math of Investment and Finance, (10) Differential Calculus, (11) Differential Calculus Advanced Topics and Applications, (12) Multivariate Differential Calculus, (13) Integral Calculus, (14) Probability Theory, and (15) Probability Distributions. (MP)

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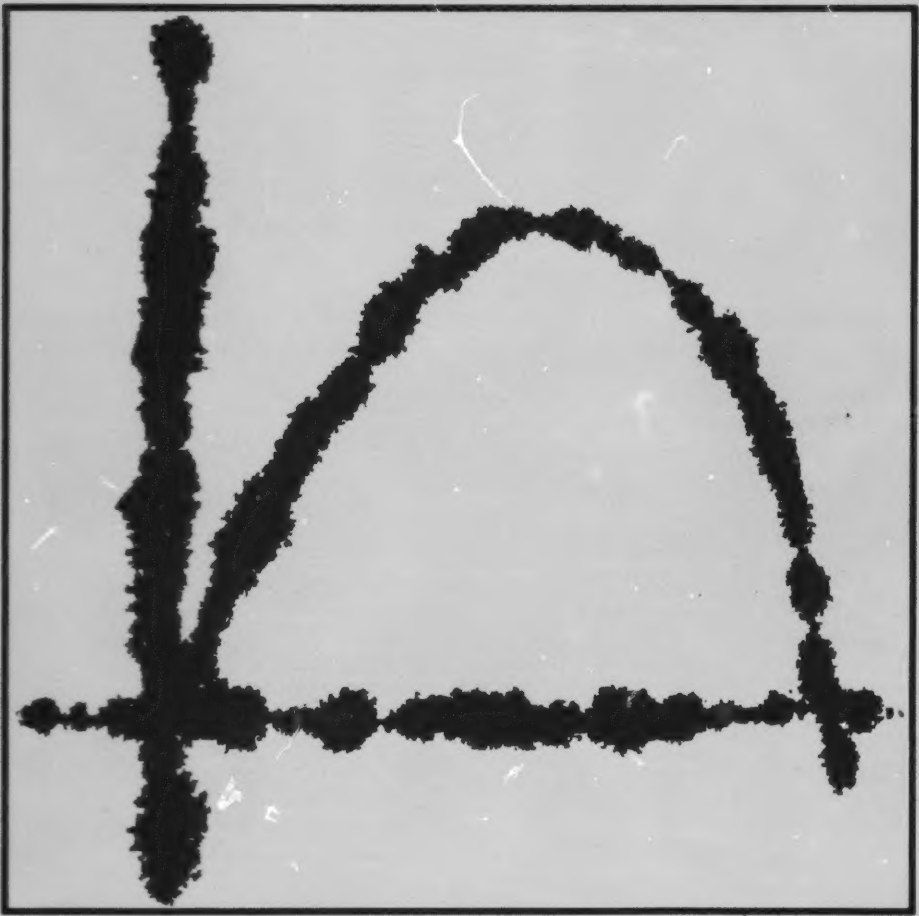
INSTRUCTOR'S GUIDE

MATHEMATICS:
FOUNDATIONS FOR BUSINESS

MANFRED W. HOPFE
LARRY BIANCI-ROSSI



SE 033 071

INSTRUCTOR'S GUIDE

MATHEMATICS: FOUNDATIONS FOR BUSINESS

Manfred W. Hopfe

Larry Bianci-Rossi

California State University, Sacremento

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TEACHING NOTES

Based on over twelve years of teaching experience, I believe the lack of algebraic comprehension is the main stumbling block to learning the topics expected of a college student in a quantitative methods course.

"A Little Calculating and a Lot of Terror Equal Math Anxiety" appeared on page one of the Wall Street Journal (March 13, 1978). The article stated that "nobody knows just how many people fear math, but people who do fear it say they have a handicap that has cut deeply into their self-esteem and their ability to cope with life." They generally can't even use pocket calculators because they aren't sure what concepts to apply in a given circumstance.

"The condition is of 'major concern,' because so many students show extreme symptoms that persist throughout life," says Alfred Wilcox, executive director of the Mathematical Association of America.

We, in business and public administration, are faced with an overabundance of adults suffering with math anxiety. I wrote this text in an attempt to provide an instructional tool to help us in our life's work of providing the encouragement necessary for our students to learn the basic requirements for a "decision scientist." Our students, for the most part, are verbally mature adults and quantitative preschoolers.

As you look through the text, you will find that the material is presented in a building-block fashion. The skills learned in earlier chapters provide a smooth transition into the more difficult topics. Chapter 0 presents a thorough review of algebra. Chapter 1 discusses systems of linear equations and sets the stage for the discussions of matrix algebra and applications in Chapters 2 and 3. Chapters 4 and 5 explain linear programming concepts and demonstrate the application of matrix inversion in the simplex algorithm.

The transition to nonlinear functions occurs in Chapters 6 and 7. Applications of nonlinear functions occur in Chapters 6 and 7. Applications of nonlinear functions are introduced in Chapter 8 by discussing financial concepts. Chapter 9 is a classical differential calculus format with applications discussed in Chapter 10. Partials are presented in an applications-oriented manner in Chapter 11. Integral calculus is presented in a classical format in Chapter 12, with Chapter 13 stressing applications. Probability concepts are discussed in Chapters 14 and 15.

For undergraduates, the following two semester approach is recommended: Semester I Chapters 0, 1, 2, 3, 4, 5, 8, and 14; and Semester II Chapters 0, 6, 7, 9, 10, 11, 12, 13, and 15. For MBAs, the entire text can be covered in one semester by stressing those topics your curriculum requires.

The Study Guide will greatly enhance learning. The text "speaks to and with the reader." Concepts are reinforced by following the text rather closely. Occasionally, the approach in the Study Guide uses the tactic of "what was meant by the text was" to enhance comprehension.

Before preparing chapter lecture notes, please read the introduction to the chapter in this instructor's manual. I have attempted to offer insights.

I regret the necessity for an errata sheet. Having worked many of the problems while on "vacation" in December may account for the majority of the needed corrections. If you find other errors, please contact me. Your letter will be answered.

As stated in the preface, I hope you find this book worthy of your time and efforts.

CHAPTER 0

A review of algebra may help dispel some of the anxiety that your students have about math. Of course, it depends on their backgrounds--however, I have found that even a cursory coverage of this material benefits most of the students.

With the influx of "returning" students, I find the time well spent in reviewing all of the topics. This is true even if you just talk about the topics. The Study Guide is keyed to the text and is an excellent learning aid.

Exercise 0.3, pp. 10-11

(Answers to problems 1 through 8 appear in problem set.)

9. $\{1, 4, 5, 7, 9, 10\}$

10. $\{7, 8\}$

11. $\{1, 4, 7, 10\}$

12. $\{1, 4\}$

13. $\{1, 5, 10, 11\}$

14. $\{4\}$

15. $\{1, 4, 5, 7, 8, 9, 10, 11\}$

16. (a) 100
    (b) 450
    (c) 150
    (d) 400

17. Six are eligible.

Exercise 0.9, pp. 30-31

1. $-a + b - 2c$

2. $x + 2y + 5$

3. $6q + 4r - 4t$

4. $7x - 3y$

5. $x^2 + x + 3$

6. $-10x + 8y + 4z$

7. $15x^2 + xy - 6y^2$

8. $-2x^2 + 13xy - 4xz - 21y^2 + 14yz$

9. $25x^2 - 40xy + 16y^2$

10. $2x - 4$

11.

$$\begin{array}{r|l}
 & 2x \;\; + 6 \\ \hline
3x - 5 & 6x^2 + \;\; 8x - 30 \\
 & \underline{6x^2 - 10x} \\
 & \qquad\quad 18x - 30 \\
 & \qquad\quad \underline{18x - 30} \\
 & \qquad\qquad\qquad 0
\end{array}$$

$(2x + 6)$

12.

$$\begin{array}{r|l}
 & 3x^2 + 0x + 5 \\ \hline
2x + 1 & 6x^3 - 3x^2 + 10x - 5 \\
 & \underline{6x^3 - 3x} \\
 & \qquad 0 \\
 & \qquad \underline{0 \qquad\qquad\qquad} \\
 & \qquad\qquad\quad 10x - 5 \\
 & \qquad\qquad\quad \underline{10x - 5} \\
 & \qquad\qquad\qquad\qquad 0
\end{array}$$

$(3x^2 + 5)$

13.

$$\begin{array}{r|l}
 & 3x^2 - 6x + 13 \\ \hline
x + 2 & 3x^3 + 0x^2 + \;\; x - \;\; 7 \\
 & \underline{3x^3 + 6x^2} \\
 & \qquad - 6x^2 + \;\; x \\
 & \qquad - \underline{6x^2 - 12x} \\
 & \qquad\qquad\quad 13x - \;\; 7 \\
 & \qquad\qquad\quad \underline{13x + 26} \\
 & \qquad\qquad\qquad\quad - 33
\end{array}$$

$\left(3x^2 - 6x + 13 - \frac{33}{x + 2}\right)$

14.

$$\begin{array}{r|l}
 & \quad x^3 + x^2 - 2x - 2 \\ \hline
3(x - 1) & x^4 + 0x^3 - 3x^2 + 0x \;\; + 2 \\
 & \underline{x^4 - \;\; x^3} \\
 & \qquad x^3 - 3x^2 \\
 & \qquad \underline{x^3 - \;\; x^2} \\
 & \qquad\quad - 2x^2 + 0x \\
 & \qquad\quad - \underline{2x^2 + 2x} \\
 & \qquad\qquad\quad - 2x + 2 \\
 & \qquad\qquad\quad - \underline{2x + 2} \\
 & \qquad\qquad\qquad\qquad 0
\end{array}$$

$1/3 \; (x^3 + x^2 - 2x - 2)$

or

$\frac{x^3}{3} + \frac{x^2}{3} - \frac{2x}{3} - \frac{2}{3}$

Exercise 0.10, p. 41

1. $(x + 2)(x + 3)$ $\qquad x = -2, -3$
2. $3(x + 1)(x - 2)$ $\qquad x = -1, \; 2$
3. $(x - 6)(x - 2)$ $\qquad x = \; 6, \; 2$
4. $(-x + 4)(x + 1)$ $\qquad x = 4, -1$
5. No integer factors; see problem 15.
6. $x^2 + 5x + \; (5/2)^2 = -6 + (5/2)^2 = 0$

   $(x + 5/2)^2 = -6 + 25/4 = 1/4$

   $x + 5/2 = \pm 1/2$

   $x = \pm 1/2 - 5/2 = -2, -3$ $\qquad (x + 2)(x +3)$

7. $x^2 - 4x + (-4/2)^2 = -4 + (4/2)^2 = 0$

$(x - 2)^2 = 0$

$x = \pm 0 + 2$

$x = 2, 2$

$(x - 2)(x - 2)$

8. $x^2 + \frac{6}{4}x + \frac{1}{4} = 0$

$x^2 + 3/2x + (3/4)^2 = -1/4 + (3/4)^2$

$(x + 3/4)^2 = 5/16$

$x + 3/4 = \pm 2.236/4$

$x = -3/4 \pm 2.236/4$

$x = -1.309, 0.1910$

$(x + 1.309)(x - 0.1910)$

9. $3(\frac{x^2}{3} - 2x + 3) = x^2 - 6x + 9 = 0$

$x^2 - 6x + (-\frac{6}{2})^2 = -9 + (-\frac{6}{2})^2$

$(x - 3)^2 = 0$

$x - 3 = \pm 0$

$x = \pm 0 + 3$

$x = 3, 3$

$1/3(x - 3)(x - 3)$

10. $x^2 - 2/3x + 1/3 = 0$

$x^2 -2/3x + (1/3)^2 = -1/3 + (1/3)^2$

$(x - 1/3)^2 = -2/9$

$x - 1/3 = \pm\sqrt{-2/9} = \pm 1/3\sqrt{-2}$

$x = 1/3 \pm 0.47140i$

$x = +0.80474, -0.13807$

$(x - 4.5756i)(x + 3.9090i)$

11. $x^2 + 5x + 6 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{25 - 4(1)(6)}}{2} = \frac{-5 \quad 1}{2} = -3, -2$$

$(x + 3)(x + 2)$

12. $3x^2 + 18x - 27 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-18 \pm \sqrt{18^2 - 4(3)(-27)}}{2(3)}$$

$$= \frac{-18 \pm \sqrt{648}}{6}$$

$$= \frac{-18 \pm 25.4558}{6}$$

$$= -7.2426, +1.2426$$

$(x + 7.2426)(x - 1.2426)$

13. $$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-2)(5)}}{2(-2)}$$

$$= \frac{1 \pm \sqrt{41}}{-4}$$

$$= \frac{1 \pm 6.403}{-4}$$

$$= -1.8508, +1.3508$$

$(x + 1.8508)(x - 1.3508$

14. $$x = \frac{-10 \pm \sqrt{10^2 -4(1)(25)}}{2(1)}$$

$$= \frac{-10 \pm \sqrt{0}}{2}$$

$$= -5, -5$$

$(x + 5)(x + 5)$

15. $$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(8)}}{2(1)}$$

$$= \frac{-3 \pm \sqrt{-23}}{2}$$

$$= \frac{-3 \pm 4.7958i}{2}$$

$$= -3.8979i, +0.8979i$$

Exercise 0.11, pp. 48-49

1. Factors of 40 are 1, 2, and 5.

$s = -5;\ (-5)^3 + 11(-5)^2 + 38(5) + 40 = 0$

Thus:

$$\begin{array}{r|l} & x^2 + 6x + 8 \\ \cline{2-2} x + 5 & x^3 + 11x^2 + 38x + 40 \\ & \underline{x^3 + 5x^2} \\ & \qquad 6x^2 + 38x \\ & \qquad \underline{6x^2 + 30x} \\ & \qquad\qquad 8x + 40 \\ & \qquad\qquad \underline{8x + 40} \\ & \qquad\qquad\qquad 0 \end{array}$$

and, $x^2 + 6x + 8 = (x + 4)(x + 2)$

therefore, $x = -2, -4, -5$

$$\begin{array}{r|rrrr} & 1 & 11 & 38 & 40 \\ -\underline{5} & & -5 & -30 & -40 \\ \hline -\underline{2} & 1 & 6 & 8 & 0 \\ & & -2 & -8 & \\ \hline & 1 & 4 & 0 & \end{array}$$

$(x + 2)(x + 4)(x + 5) = y$

2. Factors of -15 are 1, 3, 5.

$s = 5;\ (5)^3 - 9(5)^2 + 23(5) - 15 = 0$

$$\begin{array}{r|l} & x^2 - 4x + 3 \\ \cline{2-2} x - 5 & x^3 - 9x^2 + 23x - 15 \\ & \underline{x^3 - 5x^2} \\ & \quad - 4x^2 + 23x \\ & \quad \underline{- 4x^2 + 20x} \\ & \qquad\qquad 3x - 15 \\ & \qquad\qquad \underline{3x - 15} \\ & \qquad\qquad\qquad 0 \end{array}$$

and, $(x^2 - 4x + 3) = (x - 3)(x - 1)$

therefore, $x = 1, 3, 5$.

$$\begin{array}{r|rrrr} & 1 & -9 & +23 & -15 \\ \underline{5} & & 5 & -20 & +15 \\ \hline & 1 & -4 & +3 & 0 \\ \underline{3} & & 3 & -3 & \\ \hline & 1 & -1 & 0 & \end{array}$$

$(x - 1)(x - 3)(x - 5) = y$

3. Factors of -20 are 1, 2, 5.

$s = +1;\ (+1)^3 + 11(+1)^2 + 8(+1) - 20 = 0$

$$
\begin{array}{rl}
 & x^2 + 12x + 20 \\
x - 1 \mid & \overline{x^3 + 11x^2 + 8x - 20} \\
 & \underline{x^3 \qquad x^2} \\
 & \qquad 12x^2 + 8x \\
 & \qquad \underline{12x^2 - 12x} \\
 & \qquad\qquad\quad 20x - 20 \\
 & \qquad\qquad\quad \underline{20x - 20} \\
 & \qquad\qquad\qquad\quad 0
\end{array}
$$

and $(x^2 + 12x + 20) = (x + 2)(x + 10)$

therefore, $x = 1, -2, -10$.

$$
\begin{array}{r|rrrr}
 & 1 & 11 & 8 & -20 \\
\underline{+1} & & +1 & +12 & +20 \\
\hline
 & 1 & 12 & 20 & 0 \\
\underline{-2} & & -2 & -20 & \\
\hline
 & 1 & 10 & 0 &
\end{array}
$$

$(x - 1)(x + 2)(x + 10) = y$

4. Factors of -6 are 1, 2, 3.

$s = -1;\ (-1)^3 - 7(-1) - 6 = 0$

$$
\begin{array}{rl}
 & x^2 - x - 6 \\
x + 1 \mid & \overline{x^3 + 0x^2 - 7x - 6} \\
 & \underline{x^3 + x^2} \\
 & \quad - x^2 - 7x \\
 & \quad \underline{- x^2 - x} \\
 & \qquad\quad - 6x - 6 \\
 & \qquad\quad \underline{- 6x - 6} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

and $(x^2 - x - 6) = (x - 3)(x + 2)$

therefore, $x = -1, -2, 3$.

$$
\begin{array}{r|rrrr}
 & 1 & 0 & -7 & -6 \\
\underline{-1} & & -1 & +1 & +6 \\
\hline
 & 1 & -1 & -6 & 0 \\
\underline{-2} & & -2 & +6 & \\
\hline
 & 1 & -3 & 0 &
\end{array}
$$

$(x + 1)(x + 2)(x - 3) = y$

5. Factors of 12 are 1, 2, 3

$s = -1;\ (-1)^3 + 7(-1)^2 + 18(-1) + 12 = 0$

$$
\begin{array}{r|l}
 & x^2 + 6x + 12 \\
\cline{2-2}
x + 1 & x^3 + 7x^2 + 18x + 12 \\
 & \underline{x^3 + x^2} \\
 & \quad 6x^2 + 18x \\
 & \quad \underline{6x^2 + 6x} \\
 & \qquad 12x + 12 \\
 & \qquad \underline{12x + 12} \\
 & \qquad\qquad 0
\end{array}
$$

and for $x^2 + 6x + 12$

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(12)}}{2(1)} = \frac{-6 \pm \sqrt{-12}}{2}$$

$= -4.7321i,\ -1.2679i$

$$
\begin{array}{r|rrrr}
 & 1 & 7 & 18 & 12 \\
\underline{-1} & & -1 & -6 & -12 \\
\hline
 & 1 & 6 & 12 & 0
\end{array}
$$

$y = (x + 1)(x + 4.7321i)(x + 1.2679i)$

6. $(2x + 3y)(2x - 3y)$

7. $2x^2 - x^2 + y^2 + 3y^2 = x^2 + 4y^2$ (prime)

8. $(x^2 + xy) + (zx + zy) = x(x + y) + z(x + y) = (x + z)(x + y)$

9. $(x + 2y)(x - y)$

10. $xy(x - 3y + xy)$

11. $(x + 2)((x + 3)$

12. $(x + y + z)(x + y + 2z)$

Exercise 0.12, pp. 53-54

1. $\frac{5 \cdot 3}{5 \cdot 10} = \frac{3}{10}$

2. $\frac{3^3}{3^4} = 3^{3-4} = \frac{1}{3}$

3. $\frac{(x + 2)(x + 3)}{(x + 2)} = x + 3$

4. $\frac{(5x - 10)(x - 4)}{(x - 4)} = 5(x - 2)$

5. $x^{3/5 + 4/5} = x^{7/5} = \sqrt[5]{x^7}$

6. $y^{2/3 + 1/3} = y$

7. $x^{2/3 - 5/3} = x^{-1} = \frac{1}{x}$

8. $x^{2/5 - 1/5} = x^{1/5} = \sqrt[5]{x}$

9. $(x^{1/4})(y^{3/4}) = (xy^3)^{1/4}$

10. $\frac{125}{x^3} = 125x^{-3}$

11. $\frac{5}{2}(\frac{4}{4}) + \frac{7}{8} = \frac{27}{8}$

12. $\frac{3}{2}(\frac{2}{2}) - \frac{2}{4} = \frac{4}{4} = 1$

13. $\frac{2}{7}(\frac{3}{3}) + \frac{1}{3}(\frac{7}{7}) = \frac{13}{21}$

14. $x^{3/5 + 2/3} \rightarrow \frac{3}{5}(\frac{3}{3}) + \frac{2}{3}(\frac{5}{5}) = \frac{19}{15} \rightarrow x^{19/15}$

15. $x^{1/2 + 2/7} \rightarrow \frac{1}{2}(\frac{7}{7}) + \frac{2}{7}(\frac{2}{2}) = \frac{11}{14} \rightarrow x^{11/14}$

16. $x^{4/5 + 3/7} \rightarrow \frac{4}{5}(\frac{7}{7}) + \frac{3}{7}(\frac{5}{5}) = \frac{43}{35} \rightarrow x^{43/35}$

17. $x^{3/5} y^{7/8}$ [or $(x^{24} y^{35})^{1/40}$]

18. $\frac{x^2(x + y) + 2x(3x)}{(2x)(x^2)} = \frac{(x^2)(x + y + 6)}{(x^2)(2x)} = \frac{x + y + 6}{2x}$

19. $\frac{(2x + y)(x + y) = (3x)(x - y)}{(x + y)(x - y)} = \frac{5x^2 + y^2}{x^2 - y^2}$

20. $\frac{y(x^2 - 2y) - 3x(x - y)}{3xy} = \frac{x^2y - 2y^2 - 3x^2 + 3xy}{3xy}$

21. $[\frac{x + y}{2x} - \frac{2y}{1}(\frac{2x}{2x})][\frac{3x^2 + 1}{5x - 5}] = \frac{(x + y - 4xy)(3x^2 + 1)}{10x(x - 1)}$

22. (a) GE: $[\frac{56}{1}(\frac{8}{8}) + \frac{3}{4}(\frac{2}{2})] - [\frac{43}{1}(\frac{8}{8}) + \frac{5}{8}] = \frac{454}{8} - \frac{349}{8} = \frac{105}{8} = 13\frac{1}{8}$

IBM: $[\frac{276}{1}(\frac{4}{4}) + \frac{1}{4}] - [\frac{234}{1}(\frac{4}{4}) + \frac{3}{4}] = \frac{1105}{4} - \frac{939}{4} - \frac{166}{4} = 41\frac{1}{2}$

(b) GE: $\frac{1}{2}(\frac{4}{4}) - \frac{1}{8} = \frac{3}{8}$

IBM: $(262 - 259) + \frac{1}{4} = 3\frac{1}{4}$

(c) GE: $[\frac{50}{1}(\frac{4}{4}) + \frac{1}{2}(\frac{2}{2}) - \frac{3}{4}] = \frac{199}{4} = 49\frac{3}{4}$

IBM: $[\frac{259}{1}(\frac{8}{8}) + \frac{5}{8}] + [\frac{2}{1}(\frac{8}{8}) + \frac{1}{8}] = \frac{2077}{8} + \frac{17}{8} = \frac{2094}{8} = 261\frac{3}{4}$

23. 6 + 8 + 8 + 6 + 7 + 9 = 44 lbs.

2 + 14 + 5 + 12 + 10 + 3 = 46 oz = 2 lbs 14 oz

44 lbs
$\underline{\text{ 2 lbs 14 oz}}$
46 lbs 14 oz total weight

$[\frac{46}{1}(\frac{16}{16}) + \frac{14}{16}](\frac{1}{6}) = \frac{750}{96}$ = 7 lbs 13 oz average weight

24. fraction of quart

$6(\frac{1}{16})(\frac{1}{2})(\frac{1}{2}) = \frac{6}{64}(\frac{60}{60}) = 360$

$15(\frac{1}{3})(\frac{1}{16})(\frac{1}{2})(\frac{1}{2}) = \frac{15}{192}(\frac{15}{15}) = 300$

$(\frac{1}{2})(\frac{1}{2})(\frac{1}{2}) = \frac{1}{8}(\frac{480}{480}) = 480$

$(\frac{1}{5})(\frac{1}{2})(\frac{1}{2}) = \frac{1}{20}(\frac{192}{192}) = 192$

$(\frac{1}{2})(\frac{1}{3})(\frac{1}{16})(\frac{1}{2})(\frac{1}{2}) = \frac{1}{384}(\frac{10}{10}) = 10$

$1(\frac{1}{3})(\frac{1}{16})(\frac{1}{2})(\frac{1}{2}) = \frac{1}{192}(\frac{20}{20}) = 20$

$(\frac{3}{4})(\frac{1}{2})(\frac{1}{2}) = \frac{3}{16}(\frac{240}{240}) = 720$

$(\frac{1}{2})(\frac{1}{3})(\frac{1}{16})(\frac{1}{2})(\frac{1}{2}) = \frac{1}{384}(\frac{10}{10}) = 10$

$(\frac{1}{4})(\frac{1}{3})(\frac{1}{16})(\frac{1}{2})(\frac{1}{2}) = \frac{1}{768}(\frac{5}{5}) = 5$

$(\frac{3}{2})(\frac{1}{2})(\frac{1}{2}) = \frac{3}{8}(\frac{480}{480}) = \underline{1440}$

3537/3840 quart

16 · 2 · 2 · 3 · 2 · 5 · 2 = 3840 LCD

You might get away with the one quart bowl, unless you cook like I do.

24. 6/64 = 0.09375

15/192 = 0.07812

1/8 = 0.12500

1/20 = 0.05000

1/384 = 0.00260

1/192 = 0.00521

3/16 = 0.18750

1/384 = 0.00260

1/768 = 0.00130

3/8 = 0.37500

0.92108 quart

Exercise 0.13, p. 58

1. $x^2 + 2xy + y^2$

2. $a^3 + 3a^2(-b) + 3a(-b)^2 + (-b)^3$

$= a^3 - 3a^2b + 3ab^2 - b^3$

3. $(.4)^3 + 3(.4)^2(.6) + 3(.4)(.6)^2 + (.6)^3$

$= .064 + .288 + .432 + .216 = 1$

4. $(1/2)^5 + 5(1/2)^4(1/2) + 10(1/2)^3(1/2)^2 + 10(1/2)^2(1/2)^3 +$

$5(1/2)(1/2)^4 + (1/2)^5$

$= .03125 + .15625 + .3125 + .3125 + .15625 + .03125$

$= 1$

5. $3^4 + 4 \cdot 3^3 \cdot x + 6 \cdot 3^2 \cdot x^2 + 4 \cdot 3 \cdot x^3 + x^4$

$= x^4 + 12x^3 + 54x^2 + 108x + 81$

6. $n = 4$, $r = 2$, $n-r = 2$, $n - r + 1 = 3$, $n - r + 2 = 4$, $r - 1 = 1$

$\frac{4}{1} \cdot x^3 \cdot y^1 = 4x^3y$

7. $n = 5$, $r = 3$, $n - r = 2$, $n - r + 1 = 3$, $n - r + 2 = 4$, $r - 1 = 2$

$\frac{5 \cdot 4}{1 \cdot 2}(.6)^3(.4)^2 = 0.3456$

8. $n = 10,\ r = 6,\ n - r = 4,\ n - r + 1 = 5,\ n - r + 2 = 6,\ r - 1 = 5$

$$\frac{10 \times 9 \times 8 \times 7 \times 6}{1 \times 2 \times 3 \times 4 \times 5}(.5)^5(.5)^5 = 252(.5)^{10}$$

$$= 0.2461$$

9. $n = 25,\ r = 4,\ n - r = 21,\ n - r + 1 = 22,\ n - r + 2 = 23,\ r - 1 = 3$

$$\frac{25 \times 24 \times 23}{1 \times 2 \times 3} \cdot 6^{22} \cdot y^3 = 2300 \cdot 6^{22} \cdot y^3$$

$$= (1.195 \cdot 10^{91})y^3$$

10. $n = 20,\ r = 18,\ n - r = 2,\ n - r + 1 = 3,\ n - r + 2 = 4,\ r - 1 = 17$

$$\frac{20\times19\times18\times17\times16\times15\times14\times13\times12\times11\times10\times9\times8\times7\times6\times5\times4}{1\times2\times3\times4\times5\times6\times7\times8\times9\times10\times11\times12\times13\times14\times15\times16\times17}(.3)^3(.7)^{17}$$

$$= 1140\ (.3)^3(.7)^{17} = 0.0716$$

End of Chapter Exercises, pp. 58-61

1. $a^{3+1/3} = a^{10/3}$
2. $(ab)^4$
3. $(2^4)^{1/2} = 2^2 = 4$
4. $(3^3)^{2/3} = 3^2 = 9$
5. $2 \cdot x^3 \cdot 4 \cdot x^5 \cdot x^{-7} = 8 \cdot x^{3 + 5 - 7} = 8x$
6. $3 \cdot x^a \cdot x^b \cdot 5 \cdot x^{-c} = 15x^{a + b - c}$
7. $x^{-1} \cdot x^{3/2} = x^{3/2 - 1} = x^{1/2} = \sqrt{x}$
8. $5^{1/2 \cdot 4} = 5^2 = 25$
9. $\frac{(8x^3)^{1/3}}{(27y^9)^{1/3}} = \frac{2^{3 \cdot 1/3} \cdot x^{3 \cdot 1/3}}{3^{3 \cdot 1/3} \cdot y^{9 \cdot 1/3}} = \frac{2x}{3y^3}$
10. $5 \cdot x^2 \cdot 2 \cdot y^3 = 10x^2y^3$
11. $(x^{1/3+1/2})^5 = x^{5/6 \cdot 5} = x^{25/6} = \sqrt[6]{x^{25}}$
12. $\frac{5}{x^{3/4}} = \frac{5}{\sqrt[4]{x^3}}$
13. $2^{3 \cdot 1/3} \cdot x^{3 \cdot 1/3} \cdot y^{12 \cdot 1/3} = 2xy^4$
14. $\frac{2}{\frac{3}{x^4}} = \frac{2x^4}{3}$

15. $27^{2/3} = 3^{3\cdot 2/3} = 3^2 = 9$

16. $\frac{1}{2^3} + \frac{1}{2^2} = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$

17. $(3y)^{1/2} \cdot x^{2\cdot 1/2} \cdot y^{3\cdot(-1/3)} = \sqrt{3y} \cdot x \cdot y^{-1} = \frac{x\sqrt{3y}}{y} = 1.732xy^{-1/2}$

18. $a^4 \cdot a^{-5} \cdot b \cdot b^{-2} \cdot c^{1/4} = a^{-1} \cdot b^{-1} \cdot \sqrt[4]{c}$

$= \frac{\sqrt[4]{c}}{ab} = a^{-1}b^{-1}c^{1/4}$

19. $\sqrt{4(-1)} = 2\sqrt{-1} = \pm 2i$

20. $\sqrt[3]{3^3 \cdot (-1)} = 3\sqrt[3]{-1} = 3(-1) = -3$

21. $x^3 \cdot x^{-4} = x^{-1} = \frac{1}{x}$

22. $\frac{5^3}{2^3 \cdot 5^3} = \frac{1}{2^3} = \frac{1}{8}$

23. $(\frac{x}{y})^7$

24. $\sqrt[5]{2^{5-3}} = 2^{15/5} = 2^3 = 8$

25. $\frac{3^{-2}}{2^{2(-2)}} \cdot \frac{14^{2(-1/2)}}{2^{6(-1/2)}} = \frac{2^4}{3^2} \cdot \frac{2^3}{7\cdot 2} = \frac{2^{4+3-1}}{3^2 \cdot 7} = \frac{64}{63} = 1.016$

26. $x^{2/3}$

27. $a^{1/2} \cdot b^{2\cdot 1/2} \cdot c^{4\cdot 1/2} \cdot d^{1/2} = (ad)^{1/2}bc^2 = a^{1/2}bc^2d^{1/2}$

28. $a^{4/3\cdot 1/2} = a^{4/6} = a^{2/3}$

29. $(1 - i)^{2\cdot 1/3} = (1 - i)^{2/3}$

30. $(y - z)^{2/5}$

31. $7x - 5x - 3y + y + z = 2x - 2y + z$

32. $x - x + 2x + 4y - y - y = 2x + 2y$

33. $2x + x - x - 4y - y - y = 2x - 6y$

34. $5x^2 - x^2 - 3x - 2x + 7 + 6 = 4x^2 - 5x + 13$

35. $2x + 2x + 3y - 2y - z - z = 4x + y - 2z$

36. $x - x + y - z = y - z$

37. $9x^2y^2 + x^2y^2 = 10x^2y^2$

38.
$$\begin{array}{llll} x & +y & & \\ x & -y & & \\ \hline x^2 & +xy & & \\ & -xy & -y^2 & \\ \hline x^2 & & -y^2 & \end{array}$$

39.
$$\begin{array}{rrrr} x^2 & +2x & -7 & \\ & x & +4 & \\ \hline x^3 & +2x^2 & -7x & \\ & +4x^2 & +8x & -28 \\ \hline x^3 & +6x^2 & +x & -28 \end{array}$$

40. $(x - 2)(x + 3) = x^2 + 3x - 2x - 6 = x^2 + x - 6$
$$\begin{array}{rrrr} x^2 & +x & -6 & \\ & x & -1 & \\ \hline x^3 & +x^2 & -6x & \\ & -x^2 & -x & +6 \\ \hline x^3 & & -7x & +6 \end{array}$$

41. $(x - 2)(x - 2) = x^2 - 4x + 4$
$$\begin{array}{rrrr} x^2 & -4x & +4 & \\ & x & +5 & \\ \hline x^3 & -4x^2 & +4x & \\ & +5x^2 & -20x & +20 \\ \hline x^3 & +x^2 & -16x & +20 \end{array}$$

42. $(x + 1)(x + 1) = x^2 + 2x + 1$
$(x + 2)(x + 2) = x^2 + 4x + 4$
$$\begin{array}{rrrrr} x^2 & +2x & +1 & & \\ x^2 & +4x & +1 & & \\ \hline x^4 & +2x^3 & +x^2 & & \\ & +4x^3 & +8x^2 & +4x & \\ & & +4x^2 & +8x & +4 \\ \hline x^4 & +6x^3 & +13x^2 & +12x & +4 \end{array}$$

43.
$$\begin{array}{l} 5x + 7 \\ 3y + 2 \\ \hline 15xy + 21y + 10x + 14 \end{array}$$

44. $(x^2 - y^2)^{1/2}$

45. $\sqrt[3]{27x^3 + 8}$ (in simplest form)

46. $(x^2 + y^2)^{1/2}$

47. $x + 2$

48.

$$\begin{array}{r|l}
 & \;x^2 - 4x \;\;+ 3 \\
\hline
x + 1 & x^3 - 3x^2 - x + 3 \\
 & \underline{x^3 + x^2} \\
 & \quad - 4x^2 - x \\
 & \quad \underline{- 4x^2 - 4x} \\
 & \qquad\qquad 3x + 3 \\
 & \qquad\qquad \underline{3x + 3} \\
 & \qquad\qquad\quad 0
\end{array}$$

or

divisor (x - 5) = x - (-1) therefore s = -1

coefficients:

$$\begin{array}{r|rrrr}
\underline{-1} & 1 & -3 & -1 & 3 \\
 & & -1 & 4 & -3 \\
\hline
 & 1 & -4 & 3 &
\end{array}$$

answer: $x \;\; - 4x + 3$

49.

$$\begin{array}{r|rrrr}
 & 1 & 10 & 31 & 30 \\
\underline{-5} & & -5 & -25 & -30 \\
\hline
 & 1 & 5 & 6 &
\end{array}$$

answer: $x^2 + 5x + 6$

50.

$$\begin{array}{r|rrrr}
 & -1 & -2 & 5 & 6 \\
\underline{2} & & -2 & -8 & -6 \\
\hline
 & -1 & -4 & -3 &
\end{array}$$

$-x^2 - 4x - 3$

51. $$\frac{x^{(2/3 + 1/6 + 1)^2}}{x^2} = x^{22/6 - 2} = x^{5/3}$$

52.

$$\begin{array}{r|l}
 & \;x^3 + x^2 + x + 1 \text{ remainder } 1 \\
\hline
x & x^4 + x^3 + x^2 + x + 1 \\
 & \underline{x^4} \\
 & \qquad x^3 \\
 & \qquad \underline{x^3} \\
 & \qquad\qquad x^2 \\
 & \qquad\qquad \underline{x^2} \\
 & \qquad\qquad\qquad x \\
 & \qquad\qquad\qquad \underline{x} \\
 & \qquad\qquad\qquad\quad 1
\end{array}$$

53.

$$\begin{array}{r|rrrr} & 1 & -6 & 11 & -6 \\ \underline{2} & & 2 & -8 & 6 \\ \hline & 1 & -4 & 3 & 0 \\ \underline{3} & & 3 & -3 & \\ \hline & 1 & -1 & 0 & \end{array}$$

Factors	Solutions
$(x - 2)$	$x = 2$
$(x - 3)$	$x = 3$
$(x - 1)$	$x = 1$

54.

$$\begin{array}{r|rrrr} & 1 & 3 & -28 & -60 \\ \underline{-6} & & -6 & 18 & 60 \\ \hline & 1 & -3 & -10 & 0 \\ \underline{-2} & & -2 & 10 & \\ \hline & 1 & -5 & 0 & \end{array}$$

$(x + 6)$	$x = -6$
$(x + 2)$	$x = -2$
$(x - 5)$	$x = 5$

55.

$$\begin{array}{r|rrrr} & 1 & 2 & -16 & -32 \\ \underline{-2} & & -2 & 0 & 32 \\ \hline & 1 & 0 & -16 & 0 \\ \underline{-4} & & -4 & 16 & \\ \hline & 1 & -4 & 0 & \end{array}$$

$(x + 2)$	$x = -2$
$(x + 4)$	$x = -4$
$(x - 4)$	$x = 4$

56. $x(x^2 + 5x + 6)$

$$\begin{array}{r|rrr} & 1 & 5 & 6 \\ \underline{-2} & & -2 & -6 \\ \hline & 1 & 3 & 0 \end{array}$$

$(x - 0)$	$x = 0$
$(x + 2)$	$x = -2$
$(x + 3)$	$x = -3$

57.

$$\begin{array}{r|rrrr} & 1 & 1 & -37 & 35 \\ \underline{5} & & 5 & 30 & -35 \\ \hline & 1 & 6 & -7 & 0 \\ \underline{-7} & & -7 & 7 & \\ \hline & 1 & -1 & 0 & \end{array}$$

$(x - 5)$	$x = 5$
$(x + 7)$	$x = -7$
$(x - 1)$	$x = 1$

58.

$$\begin{array}{r|rrrr} & 1 & 1 & -12 & -12 \\ \underline{-1} & & -1 & 0 & 12 \\ \hline & 1 & 0 & -12 & 0 \end{array}$$

$(x + 1)$	$x = -1$
$(x + 2\sqrt{3})$	$x = -2\sqrt{3}$
$(x - 2\sqrt{3})$	$x = 2\sqrt{3}$

$$x^2 - 12 = 0$$

$$x^2 = 12$$

$$x = 2\sqrt{3}$$

$$x = 3.4641$$

59. $3x(x^2 - 25) = 3x(x - 5)(x + 5)$

$(3x)$	$x = 0$
$(x - 5)$	$x = 5$
$(x + 5)$	$x = -5$

60.

$$\begin{array}{r|rrrrr} & 1 & -5 & 0 & 5 & -1 \\ \underline{-1} & & -1 & 6 & -6 & 1 \\ \hline & 1 & -6 & 6 & -1 & 0 \\ \underline{1} & & 1 & -5 & 1 & \\ \hline & 1 & -5 & 1 & 0 & \end{array}$$

$$x^2 -5x +1 = 0$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4\cdot 1\cdot 1}}{2\cdot 1}$$

$$= \frac{5 \pm \sqrt{21}}{2}$$

$$= -0.2087,\ 4.7913$$

Factors	**Solutions**
$(x + 1)$	$x = -1$
$(x - 1)$	$x = 1$
$(x - \frac{5 + \sqrt{21}}{2})$	$x = \frac{5 + \sqrt{21}}{2}$
$(x - \frac{5 - \sqrt{21}}{2})$	$x = \frac{5 - \sqrt{21}}{2}$

61. $x(x^3 + x^2 - 37x + 35)$

$$\begin{array}{r|rrrr} & 1 & 1 & -37 & 35 \\ \underline{-7} & & -7 & 42 & -35 \\ \hline & 1 & -6 & 5 & 0 \\ \underline{5} & & 5 & -5 & \\ \hline & 1 & -1 & 0 & \end{array}$$

$(x - 0)$	$x = 0$
$(x + 7)$	$x = -7$
$(x - 5)$	$x = 5$
$(x - 1)$	$x = 1$

62.

$$\begin{array}{r|rrrrrr} & 1 & 3 & -5 & -15 & 4 & 12 \\ \underline{-3} & & -3 & 0 & 15 & 0 & -12 \\ \hline & 1 & 0 & -5 & 0 & 4 & 0 \\ \underline{-1} & & -1 & 1 & 4 & -4 & \\ \hline & 1 & -1 & -4 & 4 & 0 & \\ \underline{2} & & 2 & 2 & -4 & & \\ \hline & 1 & 1 & -2 & 0 & & \\ \underline{-2} & & -2 & 2 & & & \\ \hline & 1 & -1 & 0 & & & \end{array}$$

$(x + 3)$	$x = -3$
$(x + 1)$	$x = -1$
$(x - 2)$	$x = 2$
$(x + 2)$	$x = -2$
$(x - 1)$	$x = 1$

63. $y(x - z) + x(x - z) = (x + y)(x - z)$

64. $x^2(x + y) - 2z(x + y) = (x^2 - 2z)(x + y)$

65. $xz - z^2 + xy - zy = z(x - z) + y(x - z) = (y + z)(x - z)$

66. $x^2 - (y^2 - 2yz + z^2) = x^2 - (y - z)^2 = (x + y - z)(x - y + z)$

67. $(x^2 - y^2)(x^2 + y^2) = (x - y)(x + y)(x^2 + y^2)$

68. $(4y)(2x - 3yz)$

69. $(x - y)(x + y)$

70. $y(x^2 - 2)$

71. $(3y)^2(x^2 - 2)$

72. $(x - 6)(x - 2)$

73. $x^2 + 2x + 5$

$$x = \frac{-2 \pm \sqrt{2^2 - 4\cdot1\cdot5}}{2\cdot1}$$

$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$$= -1 \pm 2i$$

$(x + 1 + 2i)(x + 1 - 2i)$

74. $(x - 4)(x - 1)$

75. $(x - \sqrt{5})(x + \sqrt{5})$ or $(x - 2.236)(x + 2.236)$

76. $(x - 5)(x - 3)$

77. $x = \frac{0 \pm \sqrt{0^2 - 4\cdot1\cdot3}}{2\cdot1}$

$$= \frac{\pm2 \sqrt{-3}}{2}$$

$$= \pm\sqrt{-3}$$

factors $(x + \sqrt{-3})(x - \sqrt{-3})$

78. $3\cdot x^2 + 2x - 7$

$$x = \frac{-2 \pm \sqrt{2^2 - 4\cdot3\cdot(-7)}}{2\cdot3}$$

$$= \frac{-2\pm \sqrt{88}}{6}$$

$$= \frac{-1\pm \sqrt{22}}{3}$$

factors $(x + \frac{1 + \sqrt{22}}{3})(x + \frac{1 - \sqrt{22}}{3})$

$= (x + 1.8968)(x - 1.2301)$

79. $(1 - x)(1 + x)$

80. $(x + 3)(x + 3)$

81. $(3x - 4)(x + 2)$

82. $(-x - 2)(x - 3)$

83. $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

84. $(5 + x)^4 = 5^4 + 4\cdot5^3x + 6\cdot5^2x^2 + 4\cdot5\cdot x^3 + x^4$

$= 625 + 500x + 150x^2 + 20x^3 + x^4$

85. $(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

86. $(p + q)^6 = p^6 + 6p^5q + 15p^4q^2 + 20p^3q^3 + 15p^2q^4 + 6p \cdot q^5 + q^6$

87. $(1/2 + x^2)^3 = (1/2)^3 + 3(1/2)^2x^2 + 3(1/2)(x^2)^2 + (x^2)^3$

$= 1/8 + 3/4x^2 + 3/2x^4 + x^6$

88. $n = 7,\ r = 5,\ n - r = 2,\ n - r + 1 = 3,\ n - r + 2 = 4,\ 5 - 1 = 4$

$\frac{7\times6\times5\times4}{1\times2\times3\times4}\ a^3b^4 = 35 \cdot a^3b^4$

89. $n = 5,\ r = 4,\ n - r = 1,\ n - r + 1 = 2,\ n - r + 2 = 3,\ r - 1 = 3$

$\frac{5\times4\times3}{1\times2\times3}\ x^2y^3 = 10 \cdot x^2y^3$

90. $n = 9,\ r = 2,\ n - r = 7,\ n - r + 1 = 8,\ n - r + 2 = 9,\ r - 1 = 1$

$\frac{9}{1}\ (x^2)^8(3)^1 = 27x^{16}$

91. $n = 10,\ r = 7,\ n - r = 3,\ n - r + 1 = 4,\ n - r + 2 = 5,\ r - 1 = 6$

$\frac{10\times9\times8\times7\times6\times5}{1\times2\times3\times4\times5\times6}\ (x^2)^4(-2y)^6$

$= (210)(x^8)(64y^6)$

$= 13440\ x^8y^6$

92. $n = 14,\ r = 3,\ n - r = 11,\ n - r + 1 = 12,\ n - r + 2 = 13,\ r - 1 = 2$

$\frac{14\times13}{1\times2}(3x^2)^{12}(5y^2)^2$

$= 1,209,028,275x^{24}y^4$

93. $\frac{3x}{3x - 2}\left(\frac{3x + 2}{3x + 2}\right) + \frac{7}{(3x - 2)(3x + 2)}$

$= \frac{9x^2 + 6x + 7}{(3x - 2)(3x + 2)}$

94. $\frac{y^2}{8y + 2}\left(\frac{y - 3}{y - 3}\right) - \frac{3y + 2}{y - 3}\left(\frac{8y + 2}{8y + 2}\right)$

$= \frac{y^2 - 3y - (24y^2 + 22y + 4)}{(8y + 2)(y - 3)}$

$= \frac{-23y^2 - 25y - 4}{(8y + 2)(y - 3)}$

95. $\left(\frac{xy + 2x}{1}\right)\left(\frac{3x^3y^2}{3x^3y^2}\right) + \left(\frac{1}{3x}\right)\left(\frac{xy}{xy}\right) - \left(\frac{x}{xy}\right)\left(\frac{3x^2y}{3x^2y}\right)$

$$= \frac{3x^4y^3 + 6x^4y^2 + xy - 3x^3y}{3x^3y^2}$$

$$= \frac{(3x^3y^2 + 6x^3y - 3x^2 + 1)}{3x^2y}$$

96. $$\frac{\frac{x^2}{1}\left(\frac{x}{x}\right) - \frac{1}{x}}{\frac{x}{1}\left(\frac{x}{x}\right) + \frac{1}{x}\left(\frac{x}{x}\right)} + x^2$$

$$= \frac{\frac{x^3 - 1}{x}}{\frac{x^2 + x + 1}{x}} + x^2$$

$$= \frac{x^3 - 1}{x^2 + x + 1} + \frac{x^2}{1}\left(\frac{x^2 + x + 1}{x^2 + x + 1}\right)$$

$$= \frac{x^4 + 2x^3 + x^2 - 1}{x^2 + x + 1}$$

$$= x^2 + x - 1$$

check:

$$\begin{array}{r|l}
 & \quad x^2 + x - 1 \\ \hline
x^2 + x + 1 & x^4 + 2x^3 + x^2 + 0x - 1 \\
 & \underline{x^4 + x^3 + x^2} \\
 & \qquad x^3 + 0x^2 + 0x \\
 & \qquad \underline{x^3 + x^2 + x} \\
 & \qquad\quad - x^2 - x - 1 \\
 & \qquad\quad \underline{- x^2 - x - 1} \\
 & \qquad\qquad\qquad 0
\end{array}$$

97. $$\frac{\frac{3}{4}}{\frac{8}{7}} = \frac{3}{4}\left(\frac{7}{8}\right) = \frac{21}{32}$$

98. $$\frac{\frac{7}{2}}{\frac{3}{5}} = \frac{7}{2}\left(\frac{5}{3}\right) = \frac{35}{6}$$

99. $\frac{3}{4}(\frac{5}{1}) + \frac{1}{6}(\frac{7}{2})$

$= \frac{15}{4} + \frac{7}{12} = \frac{15}{4}(\frac{3}{3}) + \frac{7}{12}$

$= \frac{45 + 7}{12}$

$= \frac{13}{3}$

100. Numerator:

$\frac{(x - y)}{(x + y)} + \frac{y}{(y - x)} = \frac{(x - y)}{(x + y)} - \frac{y}{(x - y)}$ -1 factored out

$= \frac{(x - y)(x - y) - y(x + y)}{(x + y)(x - y)}$ multiply by LCD

Denominator:

$1 - \frac{3y}{(x - y)} + \frac{2y}{(x + y)} = \frac{(x + y)(x - y) - 3y(x + y) + 2y\ (x - y)}{(x + y)(x - y)}$

Fraction (expanded):

$$\frac{x^2 - 2xy + y^2 - xy - y^2}{x^2 - y^2 - 3xy - 3y^2 + 2xy - 2y^2}$$

$= \frac{x^2 - 3xy}{x^2 - xy - 6y^2}$

$= \frac{x(x - 3y)}{(x + 2y)(x - 3y)}$

$= \frac{x}{x + 2y}$

CHAPTER 1

SYSTEMS OF LINEAR EQUATIONS

Systems of linear equations are used as a vehicle to stress the concept of relationships. Next, the classic approach to slope and points is introduced. Using this as a foundation, linear equations are developed.

The process developed in this chapter relies on a situational approach. After the situations are resolved, the student is introduced to the concepts of equalities and inequalities.

By developing, through examples, simultaneous equations, the need for matrix algebra is mentioned. The capstone to this chapter is model building which reinforces the concepts learned in the preceeding sections of the chapter.

Students should be encouraged to use the Study Guide in conjunction with the chapter to enhance learning.

Exercise 1.4, pp. 73-75

1. $(y - 4) = 3(x - 2)$

   $y = -2 + 3x$

   thus, $f(x) = -2 + 3x$

2. $(y - 8) = 1/2(x - 5)$

   $= 8 - 5/2 + 1/2x$

   $f(x) = 11/2 + 1/2x$

3. $(y - 0) = -2[x - (-4)]$

   $y = -8 - 2x$

   $f(x) = -8 - 2x$

4. $(y - 0) = 1(x - 0)$

   $y = x$

   $f(x) = x$

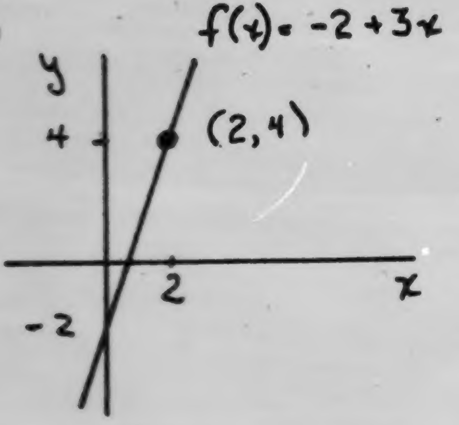
5. $(y - 3) = 0(x - 1)$

   $y = 3$

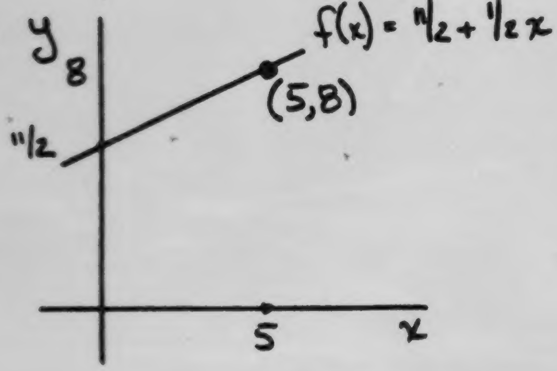
   $f(x) = 3$

6. the y-axis

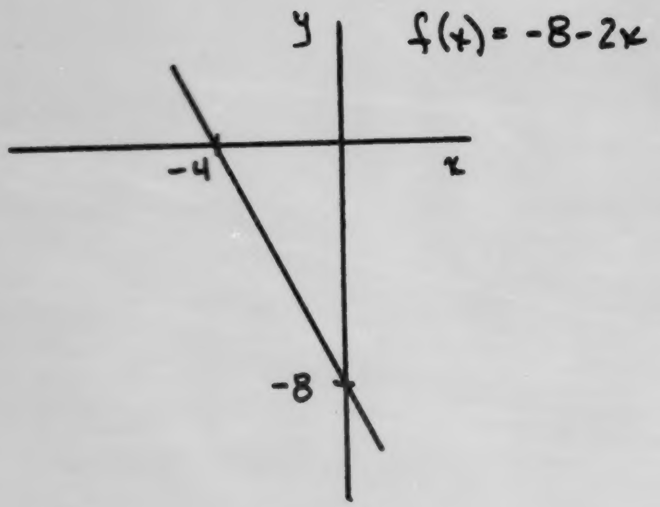
7. (1)



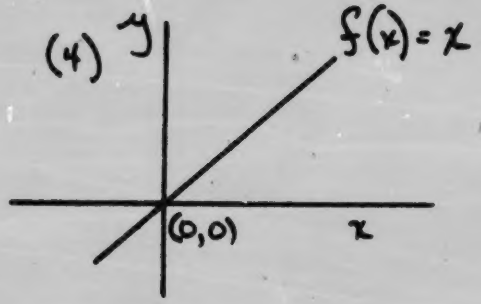
(2)



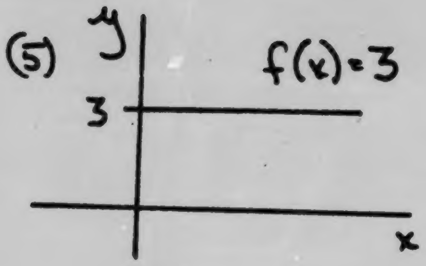
(3)



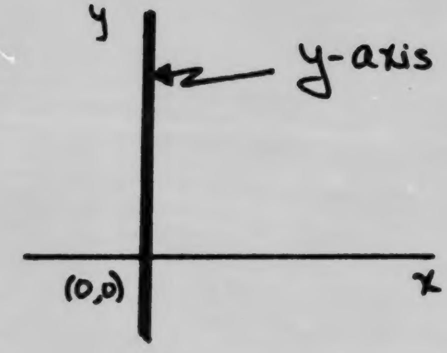
7. (4)



(5)



(6)



8. $\frac{y - 2}{x - 0} = \frac{4 - 2}{3 - 0}$

$y - 2 = 2/3x$

$y = 2 + 2/3x$

$f(x) = 2 + 2/3x$

9. $\frac{y - 3}{x - 1} = \frac{0 - 3}{0 - 1}$

$y - 3 = 3x - 3$

$y = 3x$

$f(x) = 3x$

10. $\frac{y - (-4)}{x - (-3)} = \frac{1 - (-4)}{1 - (-3)}$

$y + 4 = 5/4x + 15/4$

$y = -1/4 + 5/4x$

$f(x) = -1/4 + 5/4x$

11. $\frac{y - 1}{x - 2} = \frac{5 - 1}{-2 - 2}$

$y - 1 = -x + 2$

$y = 3 - x$

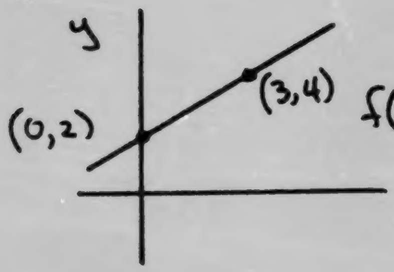
$f(x) = 3 - x$

12. $\frac{y - (-1)}{x - (-1)} = \frac{-2 - (-1)}{-5 - (-1)}$

$y + 1 = 1/4x + 1/4$

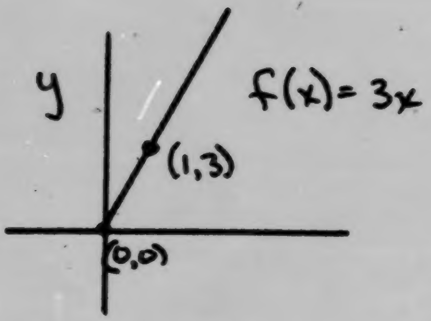
$y = -3/4 + 1/4x$

13. (8)

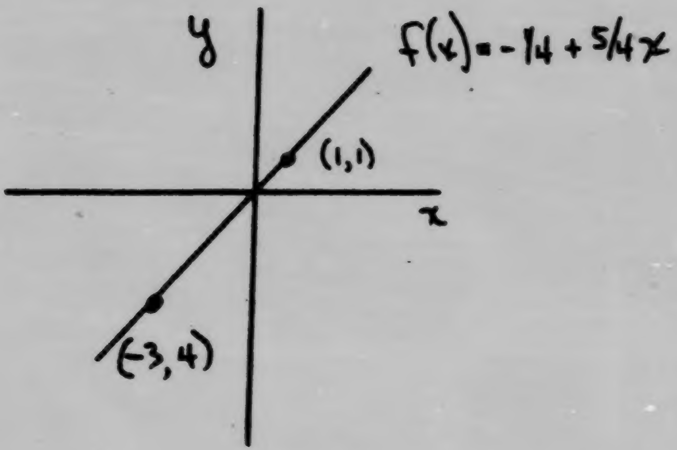


$f(x) = 2 + 2/3\,x$

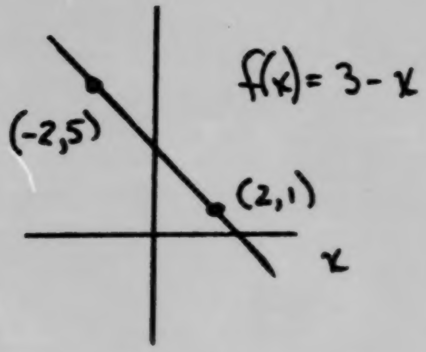
13. (9)



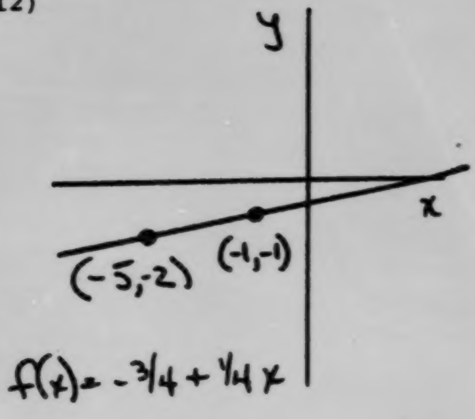
(10)



(11)



(12)



14. (a) $\frac{y - 100}{x - 167} = \frac{75 - 100}{132 - 167} = .7142857 \ldots$

$y - 100 = 0.7143x - 119.2857$

$y = -19.2857 + 0.07143x$

$f(x) = -19.2857 + 0.7143x$

(b) $f(158) = -19.2857 + 0.7143(158)$

$= 93.6$

~ 94

$f(142) = 82.1 \sim 82$

$f(124) = 69.3 \sim 69$

$f(86) = 42.1 \sim 42$

(c) $90 = -19.2857 + 0.7143x$

$x = 152.997 \sim 153$

15. one valve = \$110

four valves = \$150

$\frac{y - 110}{x - 1} = \frac{150 - 110}{4 - 1} = 13.3333$

$y - 110 = 13.333x - 13.333$

$y = 96.667 + 13.333x$

$(1 \le x \le 4) \quad (110 \le y \le 150)$

16. 16 oz = \$1.19 (pump)

20 oz = \$0.99 (refill)

$y = 1.19 + \frac{.99}{20}x$

$f(x) = 1.19 + 0.0495x$

Exercise 1.5, pp. 76-77

Problem	Slope	y-intercept
1	2	3
2	-4	1/2
3	1	-7
4	0	2
5	8	0
6	1	-4

Problem	Range	Domain	Rules
7.	11 to 17	$4 \le x \le 7$	$3 + 2x$
8.	-79/2 to +25/2	$-3 \le x \le 10$	$1/2 - 4x$
9.	-14 to -7	$0 \le x \le 7$	$-7 + x$
10.	2	$1 \le x \le 2$	2
11.	-80 to +24	$-10 \le x \le 3$	$8x$
12.	-1 to +1	$3 \le x \le 5$	$x - 4$

13. $f(x) = -2 + 4x$

14. $f(x) = 5 + 3x$

15. $(y - 3) = -2(x - 0)$

$$y = 3 - 2x$$

$$f(x) = 3 - 2x$$

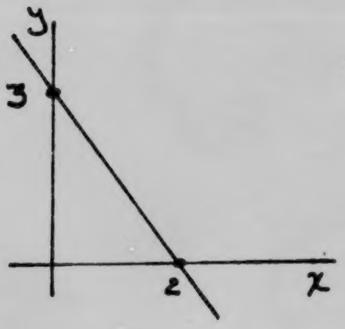
16. $f(x) = 4$

17. $f(x) = 3x$

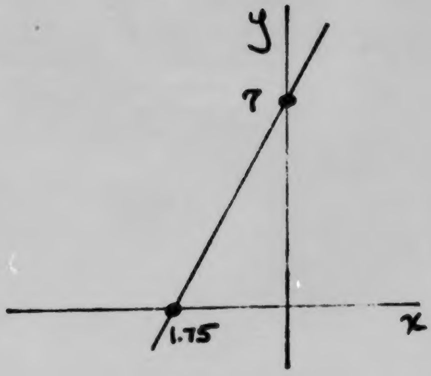
18. $f(x) = 3/4 - 1/2x$

Exercise 1.7, pp. 87-88

1.

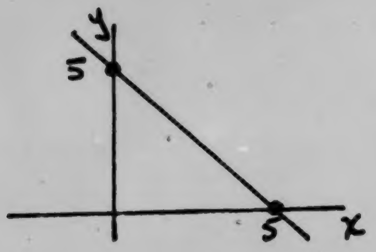


2.

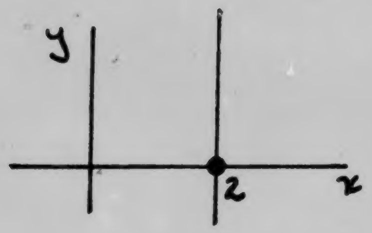


35

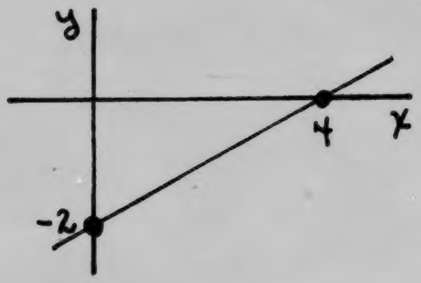
3.


y
5
5
x

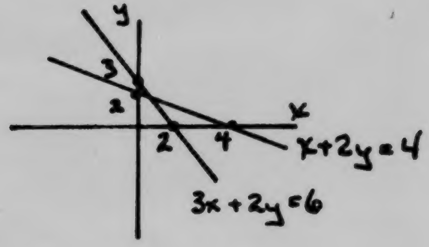
4.


y
2
x

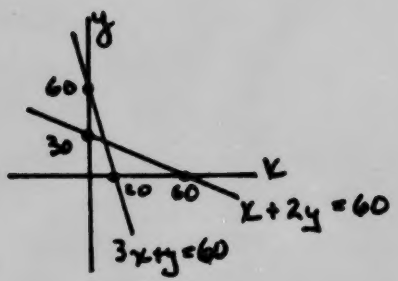
5.


y
4
x
-2

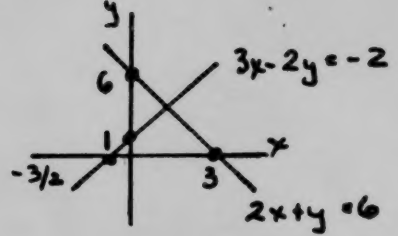
6.


y
3
2
x
2
4
x+2y=4
3x+2y=6

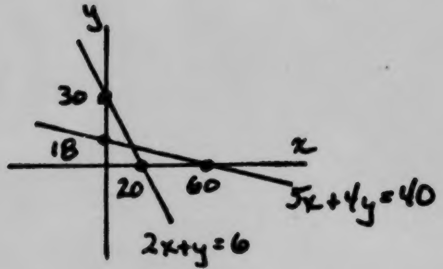
7.


y
60
30
x
20
60
x+2y=60
3x+y=60

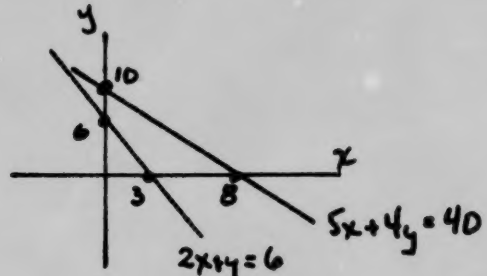
8.


y
3x-2y=-2
6
1
x
-3/2
3
2x+y=6

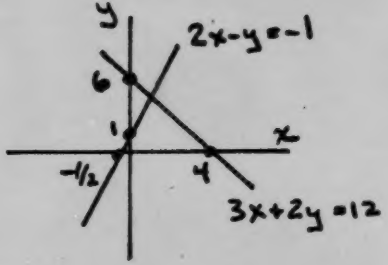
9.


y
30
18
x
20
60
5x+4y=40
2x+y=6

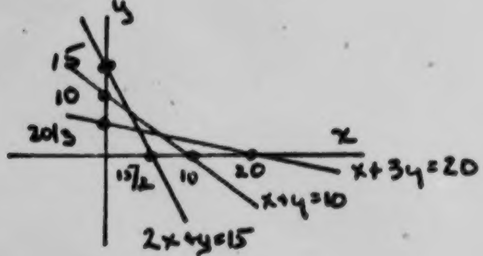
10.


y
10
6
x
3
8
5x+4y=40
2x+y=6

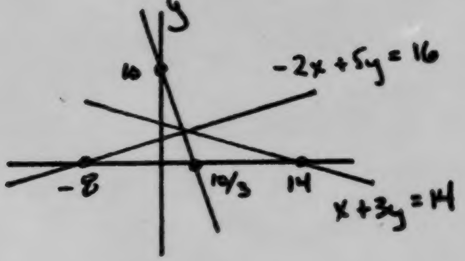
11.


y
x
6
1
-1/2
4
2x-y=-1
3x+2y=12

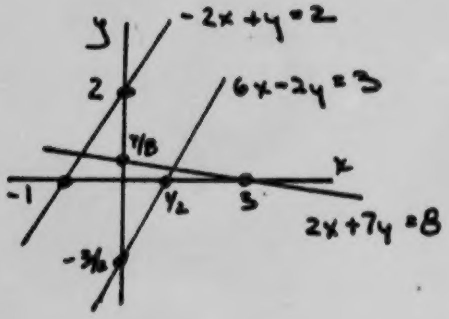
12.


y
x
15
10
20/3
15/2
10
20
x+3y=20
x+y=10
2x+y=15

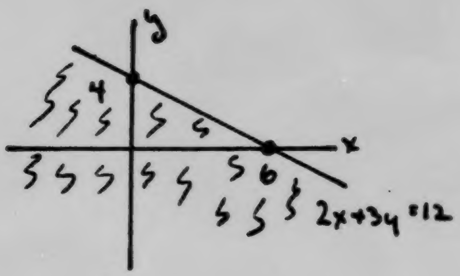
13.


y
10
-8
10/3
14
-2x+5y=16
x+3y=14

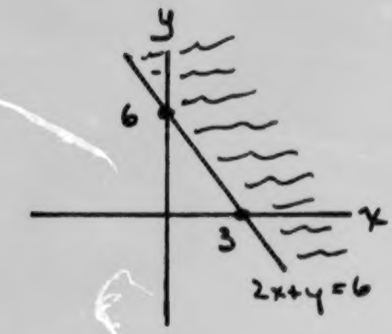
14.


y
x
2
8/7
-1
1/2
3
-3/2
-2x+y=2
6x-2y=3
2x+7y=8

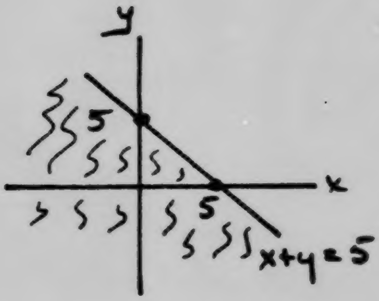
15.


y
x
4
6
2x+3y=12

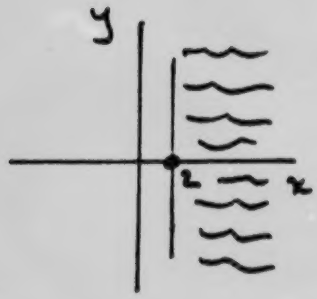
16.



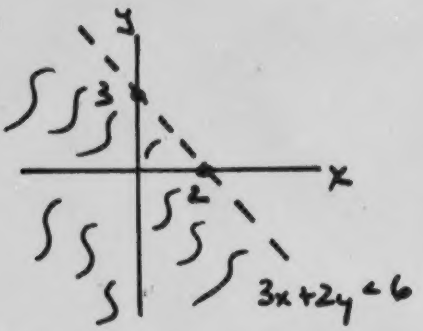
17.



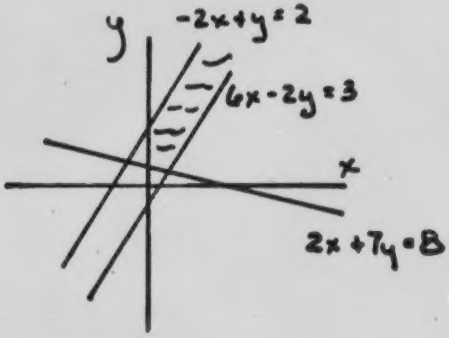
18.



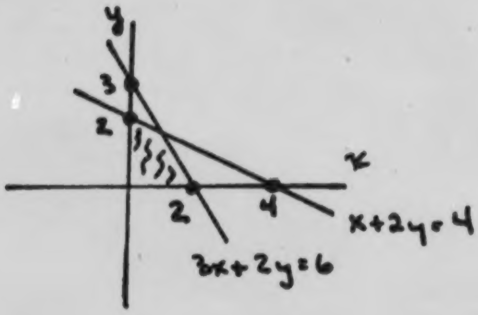
19.



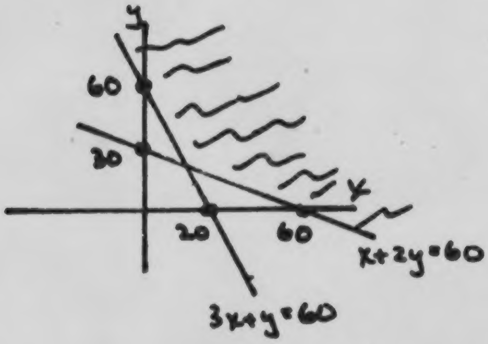
20.



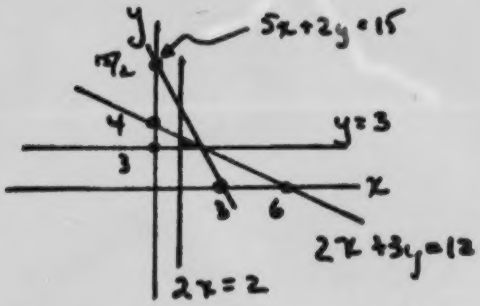
21.



22.



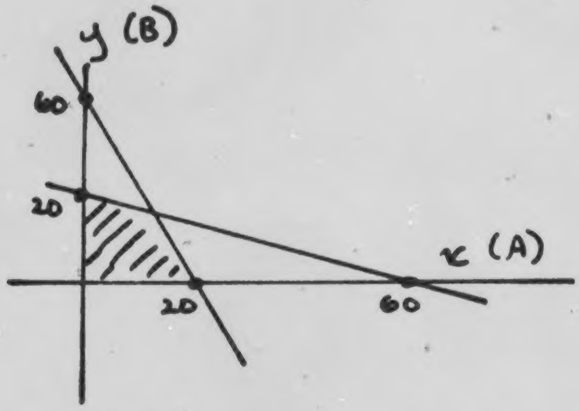
23.



24. PRODUCT

A		B		RESOURCES
3A	+	1B	≤ 60 hrs (machine time)	≤
1A	+	2B	≤ 60 hrs (packaging time)	≤

24.



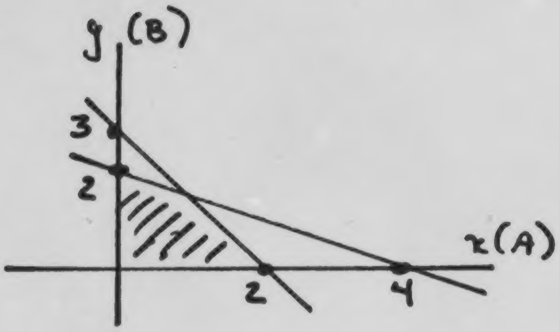
25. FORM

A B

$3A + 2B \le 6$ hrs (Center 1)

$1A + 2B \le 4$ hrs (Center 2)

RESOURCES



Exercise 1.8, pp. 93-94.

Inconsistent: Lines have no point in common.
Redundant: Equations are different representations of the same line; infinite number of solutions.

1. $3x + 2y = 6$

$x + 2y = 4$

$2x = 2$

$x = 1$

substituting $x = 1$ into second equation:

$(1) + 2y = 4$

$y = 3/2$

Thus, solution is (1, 3/2)

2. $x + y = 10$

$2x + y = 15$

$x + 3y = 20$

$3x + 2y = 25$

$x + 3y = 20$

$3x + 2y = 25$

(*3) $3x + 9y = 60$

$-7y = -35$

$y = 5$

substituting $y = 5$ into first equation:

$x + 5 = 10$

$x = 5$

check:

$5 + 5 = 10$

$2(5) + 5 = 15$

$5 + 3(5) = 15$

Thus, solution is (5,5).

3. $6x - 2y = 3$

$-2x + y = 2$

$2x + 7y = 8$

$6x - 2y = 3$

$8y = 8$

$y = 1$

$6x - 2(1) = 3$

$6x = 5$

$x = 5/6$

check:

$6(5/2) - 2(1) = 3$

$-2(5/2) + 1 \neq 8$

$2(5/2) + 7(1) \neq 8$

Thus, equations are inconsistent.

4. $3x + 2y = 7$

$6x + 4y = 14$

These equations are redundant, since two times equation one equals equation two.

5. $6x + 4y = 120$

$3x + 10y = 180$

$6x + 4y = 120$

(*2) $6x + 20y = 360$

$-16y = -240$

$y = 15$

Substituting $y = 15$ into the first equation:

$6x + 4(15) = 120$

$6x = 60$

$x = 10$

Thus, solution is (10,15).

6. $-x + 2y = 16$

$2x + y = 2$

$-2x + 4y = 32$

$2x + y = 2$

$5y = 34$

$y = 34/5$

$2x + 34/5 = 2$

$x = -12/5$

Solution (-12/5, 35/5)

7. $x + y = 4$

$2x + y = 6$

(*2) $2x + 2y = 8$

$2x + y = 6$

$y = 2$

$x + 2 = 4$

$x = 2$

Solution: (2, 2)

8. $2x + y = 9$

$x - y = 3$

$x + 2y = 6$

$3x = 12$

$x = 4$

$x + 2y = 6$

$y = 1$

Check:

$2(4) + 1 = 9$

$4 - 1 = 3$

$4 + 2(1) = 6$

Thus, solution is (4, 1)

9. $x + y = 5$

$x + z = 2$

$y + z = 1$

$y - z = 3$

$y + z = 1$

$2y = 4$

$y = 2$

$x + (2) = 5$

$x = 3$

check:

$(3) + (2) = 5$

$(3) + (-1) = 2$

$(2) + (-1) = 1$

Thus, solution (3, 2, -1)

$(3) + z = 2$

$z = -1$

$y + (-1) = 1$

$y = 2$

10. $2x + 3y - z = 5$

$x - y + z = 1$

$4x + 3y - 2z = 1$

$3x + 2y = 6$ (combining equations one and two)

$6x + y = 3$ (combining equations two and three)

$3y = 9$

$y = 3$

$6x + (3) = 3$

$6x = 0$

$x = 0$

$2(0) + 3(3) - z = 5$

$z = 4$

check:

$2(0) + 3(3) - (4) = 5$

$(0) - (3) + (4) = 1$

$4(0) + 3(3) -2(4) = 1$

Thus, solution is (0, 3, 4).

11. $x + 2y + z = 3$

$2x - y - z = -1$

$x - y + 2z = 3$

$3x + y = 2$ (combining equations one and two)

$5x - 3y = 1$ (combining equations two and three)

$14x = 7$

$x = 1/2$

$3(1/2) + y = 2$

$y = 1/2$

$(1/2) + 2(1/2) + z = 3$

$z = 3/2$

Check:

$(1/2) + 2(1/2) + (3/2) = 3$

$2(1/2) - (1/2) - (3/2) = -1$

$(1/2) - (1/2) + 2(3/2) = 3$

Thus, solution is (1/2, 1/2, 3/2)

12. MODELS

RESOURCES

x		y	
$3x$	+	$2y = 40$	(cutter)
$2x$	+	$4y = 40$	(stainer)

$3x + 2y = 40$

$2x + 4y = 40$

$y = 5$

$3x + 2(5) = 40$

$x = 10$ (10, 5)

$3x + 2y = 40$

$2x + 4y = 48$

$y = 8$

$3x + 2(8) = 40$

$x = 8$ (8, 8)

13. $10x + 4y = 320$

$6x + 12y = 480$

$y = 30$

$10x + 4(30) = 320$

$x = 20$ (20, 30)

Exercise 1.9, pp. 100-102.

1. $2 = a + b(0)$

$4 = a + b(3)$

$2 = a$

$4 = a + 3b$

$4 = 2 + 3b$

$b = 2/3$

$f(x) = 2 + 2/3x$ (Note problem set 8-12 in exercise 1.4)

2. $3 = a + b(1)$

$0 = a + b(0)$

$a = 0$

$3 = 0 + b(1)$

$b = 3$

$f(x) = 3x$

3. $-4 = a + b(-3)$

$1 = a + b(1)$

$-4 = a - 3b$

$1 = a + b$

$-5 = -4b$

$b = 5/4$

$1 = a + 5/4$

$a = -1/4$

$f(x) = -1/4 + 5/4x$

4. $1 = a + b(2)$

$5 = a + b\ (-2)$

$1 = a + 2b$

$5 = a - 2b$

$6 = 2a$

$a = 3$

$1 = 3 + 2b$

$b = -1$

$f(x) = 3 - x$

6. $40 = a + b(0)$

$50 = a + b(20)$

$40 = a$

$50 = 40 + 20b$

$b = 1/2$

$f(x) = \$40 + \$0.50x$

7. $6 = a + b(5)$

$4 = a + b(7)$

$6 = a + 5b$

$4 = a + 7b$

$2 = -2b$

$b = -1$

$6 = a - 5$

$a = 11$

$f(x) = 11- x$ (where x is the number of workers)

5. $-1 = a + b(-1)$

$-2 - a + b(-5)$

$-1 = a - b$

$-2 = a - 5b$

$1 = 4b$

$b' = 1/4$

$-1 = a = 1/4$

$a = -3/4$

$f(x) = -3/4 + 1/4x$

8. $a = \$50,000$

$b = \$20$

$p = \$45$

x = quantity (number of items)

Revenue = price × quantity

$R(x) = 45x$

Cost = fixed cost + variable cost

$C(x) = 50,000 + 20x$

Profit = Revenue - Cost

$P(x) = 45x - (50,000 + 20x)$

$= -50,000 + 25x$

$P(10,000) = -50,000 + 25(10,000) = \$200,000$

9. Demand → number of units sold depends on cost

$10 = a + b(200)$

$60 = a + b(50)$

$b = -1/3$

$10 = a + 200(-1/3)$

$a = 76.666+$

$D(x) = 76.667 - 1/3x$

Supply → number of units available depends on cost

$100 = a + b(200)$

$25 = a + b(50)$

$b = 1/2$

$100 = a + 200(1/2)$

$a = 0$

$S(x) = 1/2x$

Equilibrium: $D(x) = S(x)$

$76.666+ - 1/3x = 1/2x$

$x = 92$

$D(92) = 46$

$S(92) = 46$

10. $a = \$45,000$

$b = .60(15) = \$9$

$R(x) = 15x$

$C(x) = 45,000 + 9x$

$P(x) = -45,000 + 6x$

Breakeven:

$15x = 45,000 + 9x$

$6x = 45,000$

$x = 7,500$ items

11. $f(x) = 5 + 2.50x$

12. $24 = a + b(2)$

$48 = a + b(8)$

$48 = 3a$

$a = 16$

$24 = 16 + 2b$

$b = 4$

$f(x) = 16 + 4x$

13. $f(x) = 8000 + 500x$

14. $f(x) = 35{,}000 + 350x$

15. $f(x) = 100 + 5x$

16. $5 + a + b(3)$

$15 = a + b(6)$

$b = 10/3$

$5 = a + 3(10/3)$ $\quad a = -5$

$f(x) = -5 + 10/3x$

17. (11) slope = \$2.50, y-intercept = \$5

Slope indicates that for each additional unit of x, costs will be incurred at the rate of \$2.50 per unit of x.

The y-intercept indicates that \$5 of costs will be incurred without experiencing any units of x.

(12) slope = \$4, y-intercept = \$16

Slope indicates that for each additional item, costs will be incurred at the rate of \$4 per item.

The y-intercept indicates that \$16 of cost will be incurred without benefit of receiving any items.

(13) Slope = $500, y-intercept = $8,000

Slope indicates an annual raise expectation of $500.

The y-intercept indicates the base year salary of $8,000.

(14) Slope = $350, y-intercept = $35,000

Slope indicates that the value of the house will increase by $350 per month.

The y-intercept indicates the base year cost of the house is $35,000.

(15) Slope = $5, y-intercept = $100

Slope indicates the monthly payment rate at $5.

The y-intercept indicates the amount owed on the loan at the base period.

(16) Slope = 10/3, y-intercept = -5

Slope indicates the addition to productive capacity for each person employed.

The y-intercept suggests a negative effect on production when people are not actively producing, suggesting a non-specified production constraint.

18. $a = 1000$

$b = 5$

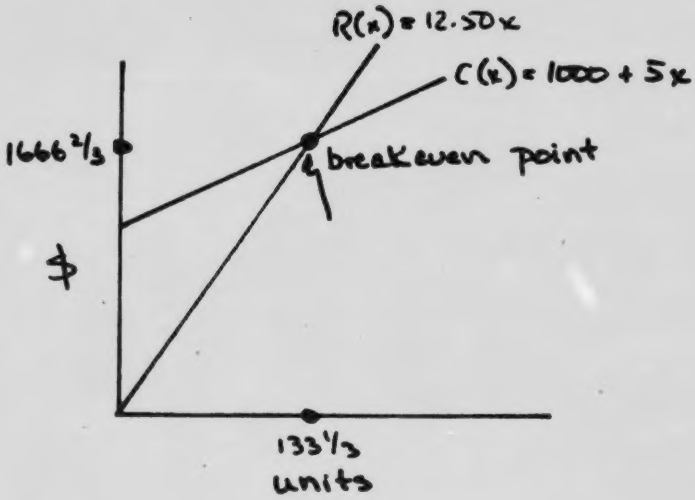
$p = 12.50$

$R(x) = 12.50x$

$C(x) = 1000 + 5x$

$12.50x = 1000 + 5x$

$x = 133.33$



19. "x" is in weeks

manual: $f(x) = 4.75(40)(12)x = 2280x$

computer: $g(x) = 200{,}000 + 1000x$

Breakeven: when manual costs = computer costs

$f(x) = g(x)$

$2280x = 200{,}000 + 1000x$

$x = 156.25$ weeks to breakeven

20. Child: $f(x) = 0.50x$ $g(x) = 5$

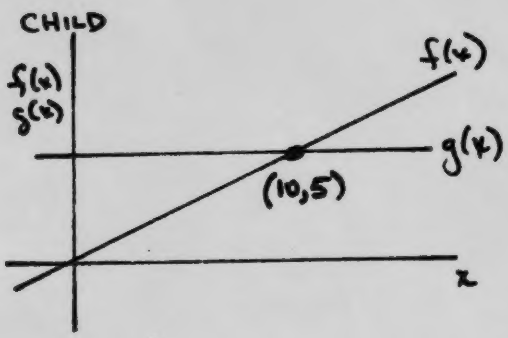
$0.50x = 5$

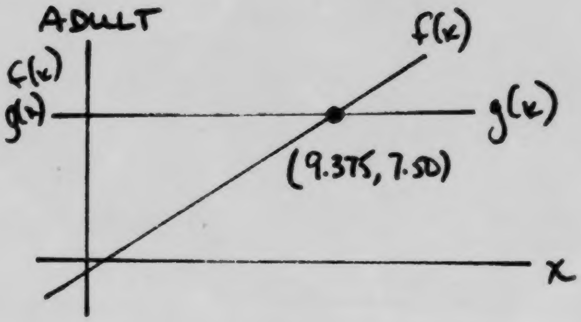
$x = 10$

Adult: $f(x) = .80x$ $g(x) = 7.50$

$0.80x = 7.50$

$x = 9.375$ (tenth time)





21. "x" is in months

lease: $f(x) = 100(36) = 3600$

buy: $g(x) = 3500 + 25x$

breakeven: $f(x) = g(x)$

$3600 = 3500 + 25x$ $x = 4$ months

CHAPTER 2

MATRIX ALGEBRA

The approach employed in presenting matrix algebra is applications oriented. With the pocket calculator available, many students will be able to calculate determinants without an understanding of matrix algebra. The challenge will be to have your students comprehend the basic concepts and to use their calculators.

Strive to prevent bogging down in the section dealing with expansion by co-factors. However, I recommend that you at least demonstrate the procedures necessary. The diagonal method will work for a 3x3 (and that's it!)--many students miss that point.

This chapter, in conjunction with Chapter 3, sets the stage for input-output analysis, Markov chains, and, eventually, the simplex algorithm.

Exercise 2.4, problem 26, on page 131 should be reviewed before assigning it. Please note my comments.

Exercise 2.2, pp. 111-112.

1. 2 x 2

2. 2 x 4

3. 5 x 2

4. 1 x 3

5. 1 x 6

6. 3 x 1

7. 1 x 1

8. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

9. 0

10. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

11. 0

12. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

13. $A^t = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$

14. $C^t = \begin{bmatrix} 3 & 2 \\ 3 & 4 \\ 2 & 0 \\ 1 & 2 \end{bmatrix}$

15. $P^t = \begin{bmatrix} 2 \\ 3 \\ 4 \\ -1 \\ 0 \\ 3 \end{bmatrix}$

16. $B^t = B$

Exercise 2.3, pp. 117-119.

1. $\begin{pmatrix} -1 & 6 & 2 \\ 9 & 3 & 9 \end{pmatrix}$

2. $\begin{pmatrix} -4 & 6 & 0 \\ 8 & 4 & 2 \end{pmatrix}$

3. $\begin{pmatrix} 3 & 0 & 2 \\ 1 & -1 & 7 \end{pmatrix}$

4. Ø (rows unequal)

5. $\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$

6. $\begin{pmatrix} -1 & 2 \\ -2 & 1 \\ 0 & -4 \end{pmatrix}$

7. $\begin{pmatrix} 5 & -5 \\ 2 & 2 \end{pmatrix}$

8. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

9. $\begin{pmatrix} 5 & 0 \\ 2 & 1 \\ 2 & 4 \end{pmatrix}$

10. (3, 1, 7)

11. Ø (columns unequal)

12. Ø (columns unequal)

13. Ø (rows unequal and columns unequal)

14. $\begin{pmatrix} -3 & 9 & 2 \\ 13 & 5 & 10 \end{pmatrix}$

15. Ø (columns unequal)

16. $\begin{pmatrix} 1 & 5 & 1 \\ 3 & 2 & 5 \end{pmatrix}$

17. $\begin{pmatrix} 3 & 6 \\ 3 & 2 \\ 3 & 8 \end{pmatrix}$

18. $\begin{pmatrix} 6 \\ 4 \\ 6 \end{pmatrix}$

19. $\begin{pmatrix} -1 & 9 \\ 6 & 3 \\ 2 & 9 \end{pmatrix}$

20. 2 × 3

21. 4 × 2

22. 0 (colums unequal; rows unequal)

23. 1 × 3

24. 2 × 2

25. 3 × 1

26. 2 × 4

27. 2 × 2

28. 1 × 3

29. 3 × 2

30. (a) $\begin{pmatrix} 3 & 6 & 5 \\ 5 & 6 & 6 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & 6 \end{pmatrix}$

31. (a) 2 × 2

(b) $\begin{pmatrix} 22 & 40 & 15 \\ 20 & 50 & 23 \end{pmatrix}$

(c) 2 × 3

Exercise 2.4, pp. 130-133.

1. $\begin{pmatrix} 0 & 14 \\ 16 & -13 \\ 2 & 1 \end{pmatrix}$

2. $\begin{pmatrix} 4 & -5 \\ 2 & 1 \end{pmatrix}$

3. $\begin{pmatrix} 3 & -6 & 12 \\ 0 & 0 & 0 \\ 2 & -4 & 8 \end{pmatrix}$

4. (20)

5. 0 $D_{col} \neq C_{row}$

6. 6

7. 0

8. $\begin{pmatrix} 2 & 6 & 4 \\ 5 & 1 & 8 \\ 1 & 3 & 2 \\ 23 & 23 & 38 \end{pmatrix}$

9. $\begin{pmatrix} 15 \\ 0 \\ 10 \end{pmatrix}$

10. $\begin{pmatrix} 7 & 12 \\ -7 & 30 \end{pmatrix}$

11. $\begin{pmatrix} -81 & -107 \\ 47 & 41 \end{pmatrix}$

12. $\begin{pmatrix} 10 & 0 \\ 0 & 5 \\ 5 & 0 \\ 15 & 20 \end{pmatrix}$

13. $\begin{pmatrix} -6 \\ 14 \end{pmatrix}$

14. $-2x_1 + 3x_2 + 0x_3$

$4x_1 + 2x_2 + 1x_3$

15. $x_1 - 2x_2 + 4x_3$

16. $\begin{pmatrix} 2 & 6 & 4 \\ 5 & 1 & 8 \\ 1 & 3 & 2 \\ 23 & 13 & 38 \end{pmatrix}$

17. 0 $C_{col} \neq F_{row}$

18. $\begin{pmatrix} 6 & -25 \\ 10 & -9 \end{pmatrix}$

19. $\emptyset$ same order, but not square

20. $\emptyset$ $F_{col} \neq x_{row}$

21. $\begin{pmatrix} 17 & 8 & -16 \\ 5 & 4 & 6 \end{pmatrix}$

22. $\begin{pmatrix} 0 & 16 & 2 \\ 14 & -13 & 1 \end{pmatrix}$

23. $\emptyset$ same order, but not square

24. (100)

25. $\begin{pmatrix} -32 & 16 \\ -8 & 4 \\ -24 & 12 \end{pmatrix}$

26. Note: This problem is a "teaser." By assigning it, anticipate reactions of "how do you work it by matrix algebra?" to "why is it located in this chapter?" I have placed several "stop and think" problems in the text.

x_1 = low risk

x_2 = high risk

x_3 = savings and loan certificate of deposit

$x_1 = 2(x_2 + x_3)$

$x_2 = 3x_3$

$x_1 + x_2 + x_3 = 12000$

$$\begin{aligned} x_1 - 2x_2 - 2x_3 &= 0 \\ x_2 - 3x_3 &= 0 \\ x_1 + x_2 + x_3 &= 12000 \end{aligned}$$

$$\begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 12000 \end{pmatrix}$$

Now you can "broach" the topic of matrix inversion which will appear in Chapter 3.

$$Ax = b$$

$$x = A^{-1}b$$

But since they can't work A^{-1}, how would the problem be solved? Substitution.

using $x_1 + x_2 + x_3 = 12000$

let $2(x_2 + x_3) + 3x_3 + x_3 = 12000$

$$2(3x_3 + x_3) + 3x_3 + x_3 = 12000$$

$$12x_3 = 12000$$

$$x_3 = 1000$$

and $x_2 = 3(1000)$

$$x_2 = 3000$$

$$x_1 = 2(3000 + 1000)$$

$$x_1 = 8000$$

check:

8000 + 3000 + 1000 = 12000

27. (a) $\begin{pmatrix} 150 & 200 & 100 \\ 250 & 100 & 300 \end{pmatrix}$

(b) $\begin{pmatrix} 375 & 400 & 200 \\ 750 & 300 & 840 \end{pmatrix}$

(c) $\begin{pmatrix} 975 \\ 1890 \end{pmatrix}$

28. $\begin{pmatrix} 25 \\ 25 \\ 20 \\ 10 \end{pmatrix} \times (2.50, 2.75, 3.00)$

$$= \begin{pmatrix} 62.50 & 68.75 & 75.00 \\ 62.50 & 68.75 & 75.00 \\ 50.00 & 55.00 & 60.00 \\ 25.00 & 27.50 & 30.00 \end{pmatrix}$$

(\$200, \$220, \$240)

(Note: Increments are \$20, or 80 × \$0.25. Here you can touch on the topic of "incremental change," which is also discussed in Chapter 3.)

29. $p = (1.98, 4.95, 12.50, 15.95, 24.98)$

$$o = \begin{pmatrix} 200 \\ 150 \\ 75 \\ 37 \\ 18 \end{pmatrix} \qquad d = (.25, .30)$$

$po = (396 + 742.50 + 937.50 + 590.15 + 449.64)$

$= (3115.79)$

$(po)d = (861.45, 934.74)$

30. Net = (827.22, 847.62, 817.50, 854.78)

31. $2x_1 + 3x_2 + x_3 = 52$

$2x_1 + 2x_2 + 2x_3 = 48$

$4x_1 + x_2 + 3x_3 = 40$

32. $A_{2\times 2} \times z_{2\times 2} = b_{2\times 1}$

$(Az)_{2\times 1} = b_{2\times 1}$

(a) $2x + 4y = 5$

$3x + y = 6$

(b) 2×1

33. $A_{3\times 3} \times x_{1\times 3} = b_{1\times 3}$

$A^t_{3\times 3} \times x^t_{3\times 1} = b^t_{3\times 1}$

Therefore:

$3x_1 + 2x_2 + x_3 = 3$

$x_1 + x_2 + 4x_3 = 2$

$x_1 + 5x_2 + x_3 = 1$

Exercise 2.5, pp. 147-150.

1. $A = \begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix}$ $|A| = (3)(1) - (1)(4) = -1$

2. $B = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$ $|B| = (3)(2) - (1)(6) = 0$

3. $C = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}$ $|C| = (5)(5) - (2)(2) = 21$

4. $F = \begin{pmatrix} -3 & 1/2 \\ -4 & 2 \end{pmatrix}$ $|F| = (-3)(2) - (1/2)(-4) = -4$

5. $A = \begin{pmatrix} 2 & 3 & 1 \\ 2 & 2 & 2 \\ 4 & 1 & 3 \end{pmatrix}$ $|A| = [(2)(2)(3) + (3)(2)(4) + (1)(1)(2)] - [(1)(2)(4) + (3)(2)(3) + (2)(1)(2)]$

$= 8$

6.
$$B = \begin{pmatrix} 4 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{aligned} |B| &= [(4)(2)(1) + (1)(1)(0) + (3)(0)(0)] - \\ &\quad [(3)(2)(0) + (1)(0)(4) + (1)(0)(1)] \\ &= 8 \end{aligned}$$

7.
$$C = \begin{pmatrix} 2 & 0 & 8 \\ 1 & 0 & 4 \\ -1 & 0 & 2 \end{pmatrix} \qquad \begin{aligned} |C| &= [(2)(0)(2) + (0)(4)(-1) + (8)(0)(1)] - \\ &\quad [(8)(0)(-1) + (4)(0)(2) + (2)(1)(0)] \\ &= 0 \end{aligned}$$

(Column 2 would indicate result.)

8.
$$D = \begin{pmatrix} -1 & 2 & -1 \\ -3 & 4 & 5 \\ -4 & 2 & 6 \end{pmatrix} \qquad \begin{aligned} |D| &= [(-1)(4)(6) + (2)(5)(-4) + (-1)(2)(3)] - \\ &\quad [(-1)(4)(-4) + (5)(2)(-1) + (6)(-3)(2)] \\ &= -28 \end{aligned}$$

9.
$$E = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 4 & 1 \end{pmatrix} \qquad |E| = \emptyset \text{ (not square)}$$

10.
$$\begin{pmatrix} 4 & -8 & 4 \\ 2 & 2 & 2 \\ -6 & 10 & -2 \end{pmatrix}$$

$$\begin{aligned} A &= (-1)^{1+1}(4)\begin{vmatrix} 2 & 2 \\ 10 & -2 \end{vmatrix} + (-1)^{1+2}(-8)\begin{vmatrix} 2 & 2 \\ -6 & -2 \end{vmatrix} \\ &\quad + (-1)^{1+3}(4)\begin{vmatrix} 2 & 2 \\ -6 & 10 \end{vmatrix} \\ &= 4(-24) + 8(8) - 4(-32) \\ &= 96 \end{aligned}$$

11.
$$B = \begin{pmatrix} 1 & 2 & 2 \\ -1 & 0 & 4 \\ 0 & 2 & 2 \end{pmatrix}$$

$$|B| = (-1)^{1+1}(1)\begin{vmatrix} 0 & 4 \\ 2 & 2 \end{vmatrix} + (-1)^{2+1}(-1)\begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} + (-1)^{3+1}(0)\begin{vmatrix} 2 & 2 \\ 0 & 4 \end{vmatrix}$$

$$= (1)(-8) + (1)0 + 0$$

$$= -8$$

12. $$C = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 0 & 1 & 1 \end{bmatrix}$$

$$|C| = (-1)^{3+1}(0)\begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix} + (-1)^{3+2}(1)\begin{vmatrix} 1 & 3 \\ 3 & 7 \end{vmatrix} + (-1)^{3+3}(1)\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix}$$

$$= 0 - 1(-2) + 1(-1)$$

$$= 1$$

13. $$D = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 2 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

$$|D| = (-1)^{1+2}(3)\begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix} + (-1)^{2+2}(2)\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} + (-1)^{3+2}(1)\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix}$$

$$= -3(-2) + 2(2) - 1(2)$$

$$= 8$$

14. $$E = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ 2 & 6 & 2 \end{pmatrix}$$

$$|E| = 0 + (-1)^{2+2}(2)\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + (-1)^{2+3}(-1)\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix}$$

$$= 0 + 2(0) + 1(0)$$

$$= 0$$

(Note: $R_3 = 2R_1$)

15.

$$F = \begin{pmatrix} 2 & 1 & 1 & 3 \\ 1 & 3 & 0 & 2 \\ 4 & 8 & -1 & 1 \\ 3 & 2 & 5 & 0 \end{pmatrix}$$

$$|F| = a_{41}F_{41} + a_{42}F_{42} + a_{43}F_{43} + a_{44}F_{44}$$

$$F_{41} = (-1)^{4+1}\begin{vmatrix} 1 & 1 & 3 \\ 3 & 0 & 2 \\ 8 & -1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -1 & -3 \\ -3 & 0 & -2 \\ -8 & 1 & -1 \end{vmatrix}$$

$$= (-3)(-1)^{2+1}\begin{vmatrix} -1 & -3 \\ 1 & -1 \end{vmatrix} + 0 + (-2)(-1)^{2+3}\begin{vmatrix} -1 & -1 \\ -8 & 1 \end{vmatrix}$$

$$= 3(4) + 0 + 2-(-9)$$

$$= -6$$

$$a_{41}F_{41} = 3(-6) = -18$$

$$F_{42} = (-1)^{4+2}\begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 4 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 4 & -1 & 1 \end{vmatrix}$$

$$= (1)(-1)^{2+1}\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} + 0 + (2)(-1)^{2+3}\begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix}$$

$$= (-1)(4) + 0 - 2(-6)$$

$$= 8$$

$$a_{42}F_{42} = 2(8) = 16$$

$$F_{43} = (-1)^{4+3}\begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 4 & 8 & 1 \end{vmatrix} = \begin{vmatrix} -2 & -1 & -3 \\ -1 & -3 & -2 \\ -4 & -8 & -1 \end{vmatrix}$$

$$= (-2)(-1)^{1+1}\begin{vmatrix} -3 & -2 \\ -8 & -1 \end{vmatrix} + (-1)(-1)^{1+2}\begin{vmatrix} -1 & -2 \\ -4 & -1 \end{vmatrix}$$

$$+ (-3)(-1)^{1+3}\begin{vmatrix} -1 & -3 \\ -4 & -8 \end{vmatrix}$$

$$= 2(-13) + 1(-7) - 3(-4) = 31$$

$$a_{43}F_{43} = 5(31)$$
$$= 155$$

$$a_{44}F_{44} = 0$$

$$|F| = -18 + 16 + 155 + 0$$
$$= 153$$

16.
$$G = \begin{pmatrix} 2 & 1 & 1 & -1 \\ 3 & 1 & 5 & 4 \\ 4 & 2 & 1 & 3 \\ 6 & 7 & 4 & 1 \end{pmatrix}$$

$$|G| = a_{11}G_{11} + a_{12}G_{12} + a_{13}G_{13} + a_{14}G_{14}$$

$$G = (-1)^{1+1} \begin{vmatrix} 1 & 5 & 4 \\ 2 & 1 & 3 \\ 7 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 4 \\ 2 & 1 & 3 \\ 7 & 4 & 1 \end{vmatrix}$$

$$= (1)(-1)^{1+1} \begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix} + (5)(-1)^{1+2} \begin{vmatrix} 2 & 3 \\ 7 & 1 \end{vmatrix} + (4)(-1)^{1+3} \begin{vmatrix} 2 & 1 \\ 7 & 4 \end{vmatrix}$$

$$= (1)(-1) - 5(-19) + 4(1)$$

$$= 88$$

$$a_{11}G_{11} = 2(88) = 176$$

$$G_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 5 & 4 \\ 4 & 1 & 3 \\ 6 & 4 & 1 \end{vmatrix} = \begin{vmatrix} -3 & -5 & -4 \\ -4 & -1 & -3 \\ -6 & -4 & -1 \end{vmatrix}$$

$$= (-3)(-1)^{1+1} \begin{vmatrix} -1 & -3 \\ -4 & -1 \end{vmatrix} + (-5)(-1)^{1+2} \begin{vmatrix} -4 & -3 \\ -6 & -1 \end{vmatrix}$$

$$+ (-4)(-1)^{1+3} \begin{vmatrix} -4 & -1 \\ -6 & -4 \end{vmatrix}$$

$$= (-3)(-11) + 5(-14) + 4(10)$$

$$= 3$$

$$a_{12}G_{12} = (1)(3) = 3$$

$$G_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 1 & 4 \\ 4 & 2 & 3 \\ 6 & 7 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 4 \\ 4 & 2 & 3 \\ 6 & 7 & 1 \end{vmatrix}$$

$$= (3)(-1)^{1+1} \begin{vmatrix} 2 & 3 \\ 7 & 1 \end{vmatrix} + (4)(-1)^{2+1} \begin{vmatrix} 1 & 4 \\ 7 & 1 \end{vmatrix}$$

$$+ (6)(-1)^{3+1} \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$$

$$= 3(-19) - 4(-27) + 6(-5)$$

$$= 21$$

$$a_{13}G_{13} = (1)(21)$$

$$= 21$$

$$G_{14} = (-1)^{1+4} \begin{vmatrix} 3 & 1 & 5 \\ 4 & 2 & 1 \\ 6 & 7 & 4 \end{vmatrix} = \begin{vmatrix} -3 & -1 & -5 \\ -4 & -2 & -1 \\ -6 & -7 & -4 \end{vmatrix}$$

$$= (-1)(-1)^{1+2} \begin{vmatrix} -4 & -1 \\ -6 & -4 \end{vmatrix} + (-2)(-1)^{2+2} \begin{vmatrix} -3 & -5 \\ -6 & -4 \end{vmatrix}$$

$$+ (-17)(-1)^{3+2} \begin{vmatrix} -3 & -5 \\ -4 & -1 \end{vmatrix}$$

$$= (1)(10) - 2(-18) + 7(-17)$$

$$= 73$$

$$a_{14}G_{14} = (-1)(-73)$$

$$= 73$$

$$|G| = 176 + 3 + 21 + 73$$

$$= 273$$

17. $|H| = 0$ (not square)

18.

$$x = \frac{\begin{vmatrix} 9 & 2 \\ 8 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}} = \frac{9 - 16}{1 - 8} = \frac{-7}{-7} = 1$$

$$y = \frac{\begin{vmatrix} 1 & 9 \\ 4 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}} = \frac{8 - 36}{1 - 8} = \frac{-28}{-7} = 4$$

(1,4)

19.

$$x = \frac{\begin{vmatrix} 14 & 3 \\ 11 & 9 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix}} = \frac{126 - 33}{18 - 18} = \text{undefined}$$

redundant

$$y = \frac{\begin{vmatrix} 2 & 14 \\ 6 & 11 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix}} = \frac{22 - 84}{18 - 18} = \text{undefined}$$

20.

$$x = \frac{\begin{vmatrix} 14 & 2 \\ 9 & 1 \end{vmatrix}}{\begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix}} = \frac{14 - 18}{-1 - 2} = \frac{-4}{-3} = \frac{4}{3}$$

$$y = \frac{\begin{vmatrix} -1 & 14 \\ 1 & 9 \end{vmatrix}}{\begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix}} = \frac{-9 - 14}{-1 - 2} = \frac{-23}{-3} = \frac{23}{3}$$

(4/3, 23/3)

21.

$$x_1 = \frac{\begin{vmatrix} 1010 & 4 & 6 \\ 670 & 4 & 5 \\ 1000 & 3 & 6 \end{vmatrix}}{\begin{vmatrix} 8 & 4 & 6 \\ 5 & 4 & 5 \\ 8 & 3 & 6 \end{vmatrix}} = \frac{56300 - 55230}{442 - 432} = \frac{1070}{10} = 107$$

$$x_2 = \frac{\begin{vmatrix} 8 & 1010 & 6 \\ 5 & 670 & 5 \\ 8 & 1000 & 6 \end{vmatrix}}{10} = \frac{102560 - 102460}{10} = \frac{100}{10} = 10$$

$$x_3 = \frac{\begin{vmatrix} 8 & 4 & 1010 \\ 5 & 4 & 670 \\ 8 & 3 & 1000 \end{vmatrix}}{10} = \frac{68590 - 68400}{10} = \frac{190}{10} = 19 \qquad (107,\ 10,\ 19)$$

22.

$$x = \frac{\begin{vmatrix} 5 & 2 \\ 6 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}} = \frac{5 - 12}{1 - 8} = \frac{-7}{-7} = 1$$

$$y = \frac{\begin{vmatrix} 1 & 5 \\ 4 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}} = \frac{6 - 20}{1 - 8} = \frac{-14}{-7} = 2 \qquad (1,\ 2)$$

23.

$$x = \frac{\begin{vmatrix} 30 & 2 \\ 18 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}} = \frac{30 - 36}{1 - 8} = \frac{-6}{-7} = \frac{6}{7}$$

$$y = \frac{\begin{vmatrix} 1 & 30 \\ 4 & 18 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}} = \frac{18 - 120}{1 - 8} = \frac{-102}{-7} = \frac{102}{7} \qquad (6/7,\ 102/7)$$

24. x_1 = daiquiris

x_2 = margaritas

$2x_1 + 2x_2 \le 3(16)$

$4x_1 + 2x_2 \le 64$

$$x_1 = \frac{\begin{vmatrix} 48 & 2 \\ 64 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix}} = \frac{96 - 128}{4 - 8} = \frac{32}{4} = 8$$

$$\begin{aligned} 2(8) + 2x_2 &= 48 \\ 2x_2 &= 30 \\ x_2 &= 15 \qquad (8,\ 15) \end{aligned}$$

(Note: $x_1 + x_2 \le 24$ is redundant if $2x_1 + 2x_2 \le 48$ is used.)

25. $p + \ell = 1000$

$0.10\,p - 0.40\ell = 0$

$$p = \frac{\begin{vmatrix} 1000 & 1 \\ 0 & -0.40 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ .10 & -.40 \end{vmatrix}} = \frac{-400}{-.5} = 800$$

$$\begin{aligned} 800 + \ell &= 1000 \\ \ell &= 200 \qquad (800,\ 200) \end{aligned}$$

26. (a) $p = 0.9p + 0.4\ell$

$p + \ell = 1000$

$$\begin{aligned} .10\,p - .40\ell &= 0 \\ p + \quad \ell &= 1000 \end{aligned}$$

(same as #25, $p = 800$ and $\ell = 200$)

(b) $\ell = .60\ell + 0.10p$

$p + \ell = 1000$

$0.40\ell - 0.10p = 0$

$p + \ell = 1000$

$-0.10p + 0.40\ell = 0$

$p + \ell = 1000$

$$p = \frac{\begin{vmatrix} 0 & 0.40 \\ 1000 & 1 \end{vmatrix}}{\begin{vmatrix} -0.10 & 0.40 \\ 1 & 1 \end{vmatrix}} = \frac{-400}{-.15} = 800$$

$p + \ell = 1000$

$\ell = 200$

(Note: This problem is setting the stage for Markov models in section 3.5.)

CHAPTER 3

MATRIX INVERSION AND APPLICATIONS

The concept of matrix inversion is not easy for many students. This chapter was written in a cookbook format to show by example how to calculate the inverse of a matrix. The emphasis on the Gaussian Elimination Method is to require the student to develop the skills needed when working the simplex algorithm.

Changing right-hand side constants, section 3.2, can be omitted if you have no plans to demonstrate sensitivity analysis when solving the simplex algorithm. I have found, however, this section provides insights to the use of an inverse. My better students are usually "turned on" by the delta method.

Input-output analysis is discussed, as are Markov models. The presentations are basic, both stressing applications of matrix algebra.

Exercise 3.1, p. 167.

1. $$\begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$3b_{11} - 2b_{21} = 1$$

$$2b_{11} + 3b_{21} = 0$$

$$-13b_{21} = 2$$

$$b_{21} = -2/13$$

(Here you need to make a decision. If you desire to have your students derive the inverse, follow the steps as shown. I prefer to derive one or two, then use the formula.)

$$13b_{11} = 3$$

$$b_{11} = 3/13$$

and $3b_{12} - 2b_{22} = 0$

$$2b_{12} + 3b_{22} = 1$$

$$-13b_{22} = -3$$

$$b_{22} = 3/13$$

$$13b_{12} = 2$$

$$b_{12} = 2/13$$

Thus,

$$A^{-1} = \begin{pmatrix} 3/13 & 2/13 \\ -2/13 & 3/13 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$$

2. $B^{-1} = 1/3 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

3. $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$2b_{11} + 3b_{21} = 1$

$1b_{11} + 4b_{21} = 0$

$-5b_{21} = 1$

$b_{21} = 1/5$

$5b_{11} = 4$

$b_{11} = 4/5$

$2b_{12} + 3b_{22} = 0$

$1b_{12} + 4b_{22} = 1$

$-5b_{22} = -2$

$b_{22} = 2/5$

$5b_{12} = -3$

$b_{12} = -3/5$

Thus,

$$C^{-1} = \begin{pmatrix} 4/5 & -3/5 \\ -1/5 & 2/5 \end{pmatrix} = 1/5 \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}$$

4. $\begin{pmatrix} 8 & 4 & 6 \\ 5 & 4 & 5 \\ 8 & 3 & 6 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$|D| = 192 + 160 + 90 - 192 - 120 - 120 = 10$

$$b_{11} = \frac{4(6) - 5(3)}{10} = 9/10$$

$$b_{21} = \frac{5(8) - 5(6)}{10} = 1$$

$$b_{31} = \frac{5(3) - 4(8)}{10} = -17/10$$

$$b_{12} = \frac{6(3) - 4(6)}{10} = -6/10$$

$$b_{22} = \frac{8(6) - 6(8)}{10} = 0$$

$$b_{32} = \frac{4(8) - 8(6)}{10} = -16/10$$

$$b_{13} = \frac{4(5) - 6(4)}{10} = -4/10$$

$$b_{23} = \frac{6(5) - 8(5)}{10} = -1$$

$$b_{33} = \frac{8(4) - 4(5)}{10} = 12/10$$

$$D^{-1} = B = \begin{pmatrix} 9/10 & -6/10 & -4/10 \\ 1 & 0 & -1 \\ -17/10 & -16/10 & 12/10 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 9 & -6 & -4 \\ 10 & 0 & -10 \\ -17 & -16 & 12 \end{pmatrix}$$

5. $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 2 \\ 3 & 4 & 2 \end{pmatrix}$

$$|A| = 0 - 6 + 0 - 0 - 8 + 4 = -10$$

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 0 & 2 \\ 4 & 2 \end{vmatrix} = -8$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} = 2$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 0 \\ 3 & 4 \end{vmatrix} = 8$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 0 \\ 4 & 2 \end{vmatrix} = 2$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} = 2$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} = -7$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} = -2$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} = -2$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = 2$$

$$A_{adj} = \begin{pmatrix} -8 & 2 & 8 \\ 2 & 2 & -7 \\ -2 & -2 & 2 \end{pmatrix}$$

$$A^{-1} = (-1/10)\begin{pmatrix} -8 & 2 & 2 \\ 2 & 2 & -2 \\ 8 & -7 & 2 \end{pmatrix}$$

6.

$$B = \begin{pmatrix} 4 & 6 & 2 \\ 2 & 2 & 8 \\ 6 & 2 & 4 \end{pmatrix}$$

$$|B| = 32 + 288 + 8 - 24 - 64 - 48 = 192$$

$$B_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 8 \\ 2 & 4 \end{vmatrix} = -8$$

$$B_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 8 \\ 6 & 4 \end{vmatrix} = 40$$

$$B_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 2 \\ 6 & 2 \end{vmatrix} = -8$$

$$B_{21} = (-1)^{2+1}\begin{vmatrix} 6 & 2 \\ 2 & 4 \end{vmatrix} = -20$$

$$B_{22} = (-1)^{2+2}\begin{vmatrix} 4 & 2 \\ 6 & 4 \end{vmatrix} = 4$$

$$B_{23} = (-1)^{2+3}\begin{vmatrix} 4 & 6 \\ 6 & 2 \end{vmatrix} = 28$$

$$B_{31} = (-1)^{3+1}\begin{vmatrix} 6 & 2 \\ 2 & 8 \end{vmatrix} = 44$$

$$B_{32} = (-1)^{3+2}\begin{vmatrix} 4 & 2 \\ 2 & 8 \end{vmatrix} = -28$$

$$B_{33} = (-1)^{3+3}\begin{vmatrix} 4 & 6 \\ 2 & 2 \end{vmatrix} = -4$$

$$B_{adj} = \begin{pmatrix} -8 & 40 & -8 \\ -20 & 4 & 28 \\ 44 & -28 & -4 \end{pmatrix}$$

$$B^{-1} = \frac{1}{192}\begin{pmatrix} -8 & -20 & 44 \\ 40 & 4 & -28 \\ -8 & 28 & -4 \end{pmatrix}$$

7. $|C| = 0$ (Note that column three is a multiple of column two.)

8. $$A = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

$$\begin{array}{cc|cc} \multicolumn{2}{c}{A} & \multicolumn{2}{c}{I} \\ 3 & 1 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{array}$$

$$\left|\begin{array}{cc|cc} 1 & 1/3 & 1/3 & 0 \\ 0 & -1/3 & -4/3 & 1 \end{array}\right|$$

$$\left|\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 4 & -3 \end{array}\right|$$

$$A^{-1} = \begin{vmatrix} -1 & 1 \\ 4 & -3 \end{vmatrix}$$

9. $$B = \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}$$

$$\begin{array}{cc|cc} \multicolumn{2}{c}{B} & \multicolumn{2}{c}{I} \\ 1 & 2 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{array}$$

$$\left|\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -7 & -4 & 1 \end{array}\right|$$

$$\left|\begin{array}{cc|cc} 1 & 0 & -1/7 & +2/7 \\ 0 & 1 & +4/7 & -1/7 \end{array}\right|$$

$$B^{-1} = \begin{pmatrix} -1/7 & +2/7 \\ +4/7 & -1/7 \end{pmatrix}$$

10. $$C = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$$

$$\begin{array}{|cc|cc|} \multicolumn{2}{c}{C} & \multicolumn{2}{c}{I} \\ -1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}$$

$$\left|\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 0 & -3 & -1 & -1 \end{array}\right|$$

$$\left|\begin{array}{cc|cc} 1 & 0 & -1/3 & 2/3 \\ 0 & 1 & 1/3 & 1/3 \end{array}\right|$$

$$C^{-1} = \begin{pmatrix} -1/3 & 2/3 \\ 1/3 & 1/3 \end{pmatrix}$$

11. $$D = \begin{pmatrix} 4 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{|ccc|ccc|} \multicolumn{3}{c}{D} & \multicolumn{3}{c}{I} \\ 4 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}$$

$$\left|\begin{array}{ccc|ccc} 1 & 1/4 & 3/4 & 1/4 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right|$$

$$\left|\begin{array}{ccc|ccc} 1 & 0 & 5/8 & 1/4 & -1/8 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right|$$

$$\left|\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & -1/8 & -5/8 \\ 0 & 1 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right|$$

$$D^{-1} = \begin{pmatrix} 1/4 & -1/8 & -5/8 \\ 0 & 1/2 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

12.

$$E = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{array}{ccc|ccc} & E & & & I & \\ 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 5 & 7 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}$$

$$\left|\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & -1 & -2 & -3 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right|$$

$$\left|\begin{array}{ccc|ccc} 1 & 0 & -1 & -5 & 2 & 0 \\ 1 & 1 & 2 & 3 & -1 & 0 \\ 0 & 0 & -1 & -3 & 1 & 1 \end{array}\right|$$

$$\left|\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & -1 \\ 0 & 1 & 0 & -3 & 1 & 2 \\ 0 & 0 & 1 & 3 & -1 & -1 \end{array}\right|$$

$$E^{-1} = \begin{pmatrix} -2 & 1 & -1 \\ -3 & 1 & 2 \\ 3 & -1 & -1 \end{pmatrix}$$

13.

$$\begin{array}{|ccc|ccc|c} & A & & & I & & b \\ 3 & 4 & 2 & 1 & 0 & 0 & 17 \\ 1 & 2 & -1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 & 0 & 1 & 5 \end{array}$$

$$\begin{array}{|ccc|ccc|c} 1 & 4/3 & 2/3 & 1/3 & 0 & 0 & 17/3 \\ 0 & 2/3 & -5/3 & -1/3 & 1 & 0 & -11/3 \\ 0 & 1 & 2 & 0 & 0 & 1 & 5 \end{array}$$

$$\begin{array}{|ccc|ccc|c} 1 & 0 & 4 & 1 & -2 & 0 & 13 \\ 0 & 1 & -5/2 & -1/2 & 3/2 & 0 & -11/2 \\ 0 & 0 & 9/2 & 1/2 & -3/2 & 1 & 21/2 \end{array}$$

$$\begin{array}{|ccc|ccc|c} 1 & 0 & 0 & 5/9 & -2/3 & -8/9 & 11/3 \\ 0 & 1 & 0 & -2/9 & 2/3 & 5/9 & 1/3 \\ 0 & 0 & 1 & 1/9 & -1/3 & 2/9 & 7/3 \\ & I & & & A^{-1} & & x \end{array}$$

(11/3, 1/3, 7/3)

14.

$$\begin{array}{ccc|ccc|c} 2 & 4 & -1 & 1 & 0 & 0 & 8 \\ 5 & 3 & -2 & 0 & 1 & 0 & -4 \\ 1 & -6 & 7 & 0 & 0 & 1 & 9 \end{array}$$

$$\begin{array}{ccc|ccc|c} 1 & 2 & -1/2 & 1/2 & 0 & 0 & 4 \\ 0 & -7 & 1/2 & -5/2 & -5 & 0 & -24 \\ 0 & 1 & 13/2 & 5/2 & 5 & 1 & 33 \end{array}$$

$$\left(\begin{array}{ccc|ccc|c} 1 & 0 & -5/14 & -3/14 & -10/7 & 0 & -20/7 \\ 0 & 1 & -1/14 & 5/14 & 5/7 & 0 & 24/7 \\ 0 & 0 & 92/14 & 30/14 & 30/7 & 1 & 207/7 \end{array}\right)$$

$$\begin{array}{ccc|ccc|c} 1 & 0 & 0 & -63/644 & -770/644 & 5/92 & -5/4 \\ 0 & 1 & 0 & 245/644 & 490/644 & 1/92 & 15/4 \\ 0 & 0 & 1 & 15/46 & 30/46 & 7/46 & 9/2 \\ \multicolumn{3}{c}{I} & \multicolumn{3}{c}{A^{-1}} & \multicolumn{1}{c}{x} \end{array}$$

$(-5/4,\ 15/4,\ 9/2)$

Exercise 3.2, pp. 174-175.

1. $$A = \begin{pmatrix} 3 & 4 & 2 \\ 1 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$|A| = 12 - 0 + 2 - 0 + 3 - 8 = 9$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 5$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = -2$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix} = -6$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} = 6$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} = -3$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 4 & 2 \\ 2 & -1 \end{vmatrix} = -8 \qquad A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} = 5$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 2$$

$$A_{adj} = \begin{pmatrix} 5 & -2 & 1 \\ -6 & 6 & -3 \\ -8 & 5 & 2 \end{pmatrix}$$

$$A^{-1} = 1/9\begin{pmatrix} 5 & -6 & -8 \\ -2 & 6 & 5 \\ 1 & -3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 5 & -6 & -8 \\ -2 & 6 & 5 \\ 1 & -3 & 2 \end{pmatrix}\begin{pmatrix} 17 \\ 2 \\ 5 \end{pmatrix} = \begin{array}{r} 33/9 \\ 3/9 \\ 21/9 \end{array}$$

$$A^{-1}b_{\Delta} + x = 1/9\begin{pmatrix} 5 & -6 & -8 \\ -2 & 6 & 5 \\ 1 & -3 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 13 \\ -1 \end{pmatrix} + \begin{pmatrix} 33/9 \\ 3/9 \\ 21/9 \end{pmatrix}$$

$$\begin{pmatrix} -45/9 \\ 63/9 \\ -36/9 \end{pmatrix} + \begin{pmatrix} 33/9 \\ 3/9 \\ 21/9 \end{pmatrix} = \begin{pmatrix} -12/9 \\ 66/9 \\ -15/9 \end{pmatrix} = \begin{pmatrix} -4/3 \\ 22/3 \\ -5/3 \end{pmatrix}$$

$$A^{-1}\ b_{new} = 1/9\begin{pmatrix} 5 & -6 & -8 \\ -2 & 6 & 5 \\ 1 & -3 & 2 \end{pmatrix}\begin{pmatrix} 22 \\ 15 \\ 4 \end{pmatrix} \quad \begin{pmatrix} -12/9 \\ 66/9 \\ -15/9 \end{pmatrix}$$

$$\begin{pmatrix} -4/3 \\ 22/3 \\ -5/3 \end{pmatrix}$$

2. $$A^{-1}b_{\text{new}} = 1/9\begin{pmatrix} 5 & -6 & -8 \\ -2 & 6 & 5 \\ 1 & -3 & 2 \end{pmatrix}\begin{pmatrix} 14 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 6/9 \\ 21/9 \\ 12/9 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 7/3 \\ 4/3 \end{pmatrix}$$

$$A^{-1}b_{\Delta} + x = 1/9\begin{pmatrix} 5 & -6 & -8 \\ -2 & 6 & 6 \\ 1 & -3 & 2 \end{pmatrix}\begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 11/3 \\ 1/3 \\ 7/3 \end{pmatrix}$$

$$= \begin{pmatrix} -27/9 \\ 18/9 \\ -9/9 \end{pmatrix} - \begin{pmatrix} 33/9 \\ 3/9 \\ 21/9 \end{pmatrix} = \begin{pmatrix} 6/9 \\ 21/9 \\ 12/9 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 7/3 \\ 4/3 \end{pmatrix}$$

3. $$A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$$

$$\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}$$

$$\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}$$

$$\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}$$

$$\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}$$

$$A^{-1}b = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ -2 \end{pmatrix}$$

4. By fixing x_2 and x_3, x_1 is literally boundless. This question is setting the stage for Chapters 4 and 5.

5. $3A + 2B + C = 28$

$2A + B + 4C = 37$

$3A + 3B + C = 33$

$$\left(\begin{array}{ccc|ccc|c} 3 & 2 & 1 & 1 & 0 & 0 & 28 \\ 3 & 1 & 4 & 0 & 1 & 0 & 37 \\ 3 & 3 & 1 & 0 & 0 & 1 & 33 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc|c} 1 & 2/3 & 1/3 & 1/3 & 0 & 0 & 28/3 \\ 0 & -1/3 & 10/3 & -2/3 & 1 & 0 & 55/3 \\ 0 & 1 & 0 & -1 & 0 & 1 & 5 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc|c} 1 & 0 & 7 & -1 & 2 & 0 & 46 \\ 0 & 1 & -10 & 2 & -3 & 0 & -55 \\ 0 & 0 & 10 & -3 & 3 & 1 & 60 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc|c} 1 & 0 & 0 & 11/10 & -1/10 & -7/10 & 4 \\ 0 & 1 & 0 & -1 & 0 & 1 & 5 \\ 0 & 0 & 1 & -3/10 & 3/10 & 1/10 & 6 \end{array}\right)$$

Monday: $A = 4,\ B = 5,\ C = 6$

Tuesday: $A = (11/10)32 + (-1/10)33 + (-7/10)37 + 4 = 10$

$B = (-1)\ 32 + (0)\ 33 + (1)\ 37 + 5 = 10$

$C = (-3/10)32 + (3/10)\ 33 + (1/10)\ 37 + 6 = 10$

Wednesday: $A = (11/10)29 + (1/10)2 - (7/10)29 + 4 = 15.4$

$B = (-1)\ 29 + (0)\ 2 + (1)\ 29 + 5 = 5$

$C = (-3/10)29 + (3/10)2 + (1/10)29 + 6 = 5$

Thursday: $A = (11/10)42 - (1/10)23 - (7/10)47 + 4 = 15$

$B = (-1) \quad 42 + (0) \quad 23 + (1) \quad 47 + 5 \quad 10$

$C = (-3/10)42 + (3/10)23 + (1/10)47 + 6 = 5$

Friday: $A = (11/10)(-18) - (1/10)(-19) - (7/10)(-21) + 4 = 0.8$

$B = (-1) \quad (-18) + (0) \quad (-19) + (1) \quad (-21) + 5 = 2$

$C = (-3/10)(-18) + (3/10 \ (-19) + (1/10)(-21) + 6 = 3.6$

Exercise 3.4, pp. 185-186.

1. $$P = \begin{pmatrix} 200/400 & 100/1000 \\ 400/400 & 500/1000 \end{pmatrix} \qquad P = \begin{pmatrix} 1/2 & 1/10 \\ 1 & 1/2 \end{pmatrix}$$

$$(I - P) = \begin{pmatrix} 1/2 & -1/10 \\ -1 & 1/2 \end{pmatrix} \qquad |I - P| = 1/4 - 1/10 = 3/20$$

$$(I - P)^{-1} = 20/3 \begin{pmatrix} 1/2 & 1/10 \\ 1 & 1/2 \end{pmatrix} = 1/3 \begin{pmatrix} 10 & 2 \\ 20 & 10 \end{pmatrix}$$

$$x = 1/3 \begin{pmatrix} 10 & 2 \\ 20 & 10 \end{pmatrix} \begin{pmatrix} 100 \\ 100 \end{pmatrix}$$

$$= 1/3 \begin{pmatrix} 1000 + 200 \\ 2000 + 1000 \end{pmatrix} = \begin{pmatrix} 400 \\ 1000 \end{pmatrix}$$

2. $$P = \begin{pmatrix} 500/1000 & 400/800 \\ 300/1000 & 200/800 \end{pmatrix} \qquad P = \begin{pmatrix} 1/2 & 1/2 \\ 3/10 & 1/4 \end{pmatrix}$$

$$(I - P) = \begin{pmatrix} 1/2 & -1/2 \\ -3/10 & 3/4 \end{pmatrix} \qquad |I - P| = 3/8 - 3/20 = 9/40$$

$$(I - P)^{-1} = 40/9 \begin{pmatrix} 3/4 & 1/2 \\ 3/10 & 1/2 \end{pmatrix} = 1/9 \begin{pmatrix} 30 & 20 \\ 12 & 20 \end{pmatrix}$$

$$x = 1/9 \begin{pmatrix} 30 & 20 \\ 12 & 20 \end{pmatrix} \begin{pmatrix} 100 \\ 300 \end{pmatrix}$$

$$= 1/9 \begin{pmatrix} 3000 + 6000 \\ 1200 + 6000 \end{pmatrix} = \begin{pmatrix} 1000 \\ 800 \end{pmatrix}$$

3. $$P = \begin{pmatrix} 5/24 & 3/13 & 1/5 \\ 1/6 & 5/13 & 1/10 \\ 1/3 & 1/13 & 2/5 \end{pmatrix}$$

$$(I - P) = \begin{pmatrix} 19/24 & -3/13 & -1/5 \\ -1/6 & 8/13 & -1/10 \\ -1/3 & -1/13 & 3/5 \end{pmatrix}$$

$$\begin{aligned} |I - P| &= (57/195 - 3/390 - 1/390) - (8/195 + 9/390 + 19/3120) \\ &= 110/390 - 219/3120 \\ &= 661/3120 \end{aligned}$$

$$\begin{pmatrix} 24/65 - 1/130 & -3/30 - 1/30 & 1/78 + 8/39 \\ -9/65 - 1/65 & 57/120 - 1/15 & -19/312 - 3/39 \\ 3/130 + 8/65 & -19/240 - 1/30 & 152/312 - 3/78 \end{pmatrix}$$

$$\begin{pmatrix} 47/130 & 4/30 & 17/78 \\ 10/65 & 49/120 & 43/312 \\ 19/130 & 27/240 & 140/312 \end{pmatrix}$$

$$(I - P)^{-1} = 3120/661 \begin{pmatrix} 47/130 & 10/65 & 19/130 \\ 4/30 & 49/120 & 27/240 \\ & 43/312 & 140/312 \end{pmatrix}$$

$$= 1/661 \begin{pmatrix} 1128 & 480 & 456 \\ 416 & 1247 & 351 \\ 680 & 430 & 1400 \end{pmatrix}$$

$$x = 1/661 \begin{pmatrix} 1128 & +480 & 456 \\ +416 & 1274 & +351 \\ 680 & +430 & 1400 \end{pmatrix} \begin{pmatrix} 125 \\ 200 \\ 350 \end{pmatrix} = \begin{pmatrix} 600 \\ 650 \\ 1000 \end{pmatrix}$$

4. $$P = \begin{pmatrix} 1/12 & 1/2 & 1/4 \\ 1/3 & 1/5 & 1/6 \\ 1/6 & 3/10 & 1/2 \end{pmatrix} \qquad (I - P) = \begin{pmatrix} 11/12 & -1/2 & -1/4 \\ -1/3 & 4/5 & -1/6 \\ -1/6 & -3/10 & 1/2 \end{pmatrix}$$

$$\begin{aligned} |I - P| &= (44/120 - 1/72 - 3/120) - (4/120 + 33/720 + 1/12) \\ &= (264/720 - 10/720 - 18/720) - (24/720 + 33/720 + 60/720) \\ &= 119/720 \end{aligned}$$

$$\begin{pmatrix} 4/10 - 3/60 & -1/6 - 1/36 & 3/30 + 4/30 \\ -1/4 - 3/40 & 11/24 - 1/24 & -33/120 - 1/12 \\ 1/12 + 4/20 & -11/72 - 1/12 & 44/60 - 1/6 \end{pmatrix}$$

$$= \begin{pmatrix} 21/60 & 7/36 & 7/30 \\ 13/40 & 10/24 & 43/120 \\ 17/60 & 17/72 & 34/60 \end{pmatrix}$$

$$(I - P)^{-1} = 720/119 \begin{pmatrix} 21/60 & 13/40 & 17/60 \\ 7/36 & 10/24 & 17/72 \\ 7/30 & 43/120 & 34/60 \end{pmatrix}$$

$$= 1/119 \begin{pmatrix} 252 & 234 & 204 \\ 140 & 300 & 170 \\ 168 & 258 & 408 \end{pmatrix}$$

$$x = 1/119 \begin{pmatrix} 252 & 234 & 204 \\ 140 & 300 & 170 \\ 168 & 258 & 408 \end{pmatrix} \begin{pmatrix} 150 \\ 100 \\ 50 \end{pmatrix} = \begin{pmatrix} 600 \\ 500 \\ 600 \end{pmatrix}$$

5. $$x = 1/3 \begin{pmatrix} 10 & 2 \\ 20 & 10 \end{pmatrix} \begin{pmatrix} 200 \\ 100 \end{pmatrix}$$

$$1/3 \begin{pmatrix} 2000 + 200 \\ 4000 + 1000 \end{pmatrix} = \begin{pmatrix} 733.33 \\ 1666.67 \end{pmatrix}$$

6. $$x = 1/3 \begin{pmatrix} 10 & 2 \\ 20 & 2 \end{pmatrix} \begin{pmatrix} 150 \\ 300 \end{pmatrix}$$

$$1/3 \begin{pmatrix} 1500 + 600 \\ 3000 + 3000 \end{pmatrix} = \begin{pmatrix} 700 \\ 2000 \end{pmatrix}$$

7. $$x = 1/9 \begin{pmatrix} 30 & 20 \\ 12 & 20 \end{pmatrix} \begin{pmatrix} 200 \\ 300 \end{pmatrix} = \begin{pmatrix} 1333.33 \\ 933.33 \end{pmatrix}$$

8. $$x = 1/661 \begin{pmatrix} 1128 & 480 & 456 \\ 416 & 1274 & 351 \\ 680 & 430 & 1400 \end{pmatrix} \begin{pmatrix} 200 \\ 300 \\ 300 \end{pmatrix} = \begin{pmatrix} 766.1 \\ 863.4 \\ 1036.3 \end{pmatrix}$$

9. $$x = 1/20 \begin{pmatrix} 45 & 45 \\ 18 & 42 \end{pmatrix} \begin{pmatrix} 300 \\ 400 \end{pmatrix} = \begin{pmatrix} 1575 \\ 1100 \end{pmatrix}$$

10. $$x = 1/20 \begin{pmatrix} 45 & 45 \\ 18 & 42 \end{pmatrix} \begin{pmatrix} 2000 \\ 3000 \end{pmatrix} = \begin{pmatrix} 11250 \\ 8100 \end{pmatrix}$$

11. $$x = 1/119 \begin{pmatrix} 252 & 234 & 204 \\ 140 & 300 & 170 \\ 168 & 258 & 408 \end{pmatrix} \begin{pmatrix} 100 \\ 100 \\ 100 \end{pmatrix} = \begin{pmatrix} 579.83 \\ 512.61 \\ 700.84 \end{pmatrix}$$

12. $$x = 1/119 \begin{pmatrix} 252 & 234 & 204 \\ 140 & 300 & 170 \\ 168 & 258 & 408 \end{pmatrix} \begin{pmatrix} 625 \\ 350 \\ 200 \end{pmatrix} = \begin{pmatrix} 2354.62 \\ 1903.36 \\ 2326.89 \end{pmatrix}$$

Exercise 3.5, pp. 193-194.

1. $$(.90,\ .10)\begin{pmatrix} .60 & .40 \\ .20 & .80 \end{pmatrix} = (.56,\ .44)$$

$$(.56,\ .44)\begin{pmatrix} .6 & .4 \\ .2 & .8 \end{pmatrix} = (.424,\ .576)$$

$$(4.24,\ .576)\begin{pmatrix} .6 & .4 \\ .2 & .8 \end{pmatrix} = (.3696,\ .6304)$$

$$(m_1,\ m_2)\begin{pmatrix} .6 & .4 \\ .2 & .8 \end{pmatrix} = (m_1,\ m_2)$$

$.6m_1 + .2m_2 = m_1$

$.4m_1 + .8m_2 = m_2$

check: $-.4m_1 + .2m_2 = 0$

$.4m_1 - .2m_2 = 0$

$$m_1 = \frac{\begin{vmatrix} 0 & .2 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} -.4 & .2 \\ 1 & 1 \end{vmatrix}} = \frac{0 - .2}{-.4 - .2} = \frac{-.2}{-.6} = .333$$

$$m_2 = \frac{\begin{vmatrix} -.4 & 0 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} -.4 & .2 \\ 1 & 1 \end{vmatrix}} = \frac{-.4 - 0}{-.4 - .2} = \frac{-.4}{-.6} = .667$$

$$(.333,\ .667)\begin{pmatrix} .6 & .4 \\ .2 & .8 \end{pmatrix} = (.333,\ .667)$$

a. 44%

b. 63.04%

c. (1/3, 2/3)

2.

From	To Vanilla	Chocolate
Vanilla	.70	.30
Chocolate	.10	.90

$$(.5,\ .5)\begin{bmatrix} .7 & .3 \\ .1 & .9 \end{bmatrix} = (.4,\ .6)$$

$$(.4,\ .6)\begin{bmatrix} .7 & .3 \\ .1 & .9 \end{bmatrix} = (.34,\ 66)$$

$$m_1,\ m_2)\begin{bmatrix} .7 & .3 \\ .1 & .9 \end{bmatrix} = (m_1,\ m_2)$$

$.7m_1 + .1m_2 = m_1$

$.3m_1 + .9m_2 = m_2$

check: $-.3m_1 + .1m_2 = 0$

$.3m_1 - .1m_2 = 0$

$$m = \frac{\begin{vmatrix} 0 & .1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} -.3 & .1 \\ 1 & 1 \end{vmatrix}} = \frac{0 - .1}{-.3 - .1} = -.1/-.4 = .25$$

$$m = \frac{\begin{vmatrix} -.3 & 0 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} -.3 & .1 \\ 1 & 1 \end{vmatrix}} = \frac{-.3 - 0}{-.3 - .1} = -.3/-.4 = .75$$

$$(.25,\ .75)\begin{bmatrix} .7 & .3 \\ .1 & .9 \end{bmatrix} = (.25,\ .75$$

a. (.34, .66)

b. (.25, .75)

3.

$$(m_1, m_2, m_3)\begin{bmatrix} .75 & .05 & .20 \\ .40 & .35 & .25 \\ .10 & .30 & .60 \end{bmatrix} = (m_1, m_2, m_3)$$

$$.75m_1 + .40m_2 + .10m_3 = m_1$$

$$.05m_1 + .35m_2 + .30m_3 = m_2$$

$$.20m_1 + .25m_2 + .60m_3 = m_3$$

$$-.25m_1 + .40m_2 + .10m_3 = 0$$

$$.05m_1 - .65m_2 + .30m_3 = 0$$

$$m_1 + m_2 + m_3 = 1$$

$$m_1 = \frac{\begin{vmatrix} 0 & .4 & .1 \\ 0 & -.65 & 3 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} -.25 & .4 & .1 \\ .05 & -.65 & .3 \\ 1 & 1 & 1 \end{vmatrix}} = \frac{.185}{.2875 + .12} = .454$$

$$m_2 = \frac{\begin{vmatrix} -.25 & 0 & .1 \\ .05 & 0 & .3 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} -.25 & .4 & .1 \\ .05 & -.65 & .3 \\ 1 & 1 & 1 \end{vmatrix}} = \frac{.08}{.4075} = .196$$

$$.454 + .196 + m_3 = 1$$

$$m_3 = .350$$

(.454, .196, .30)

Sacramento--return of 45.4%

4.

$$(.25, .40, .35)\begin{pmatrix} .40 & 0 & .60 \\ .25 & .25 & .50 \\ .33 & .33 & .34 \end{pmatrix} = (.3155, .2155, .469)$$

$$(.3155, .2155, .469)\begin{pmatrix} .40 & 0 & .60 \\ .25 & .25 & .50 \\ .33 & .33 & .34 \end{pmatrix} = (.334845, .208645, .45651$$

$$(.334845, .208645, .46561)\begin{pmatrix} .40 & 0 & .60 \\ .25 & .25 & .50 \\ .33 & .33 & .34 \end{pmatrix} = (.33673755, .20280955, .4604429)$$

$$(m_1, m_2, m_3)\begin{pmatrix} .40 & 0 & .60 \\ .25 & .25 & .50 \\ .33 & .33 & .34 \end{pmatrix} = (m_1, m_2, m_3)$$

$$\begin{aligned} .4m_1 + .25m_2 + .33m_3 &= m_1 \\ .25m_2 + .33m_3 &= m_3 \\ .6m_1 + .50m_2 + .34m_3 &= m_3 \end{aligned}$$

$$\text{check: } \begin{aligned} -.6m_1 + .25m_2 + .33m_3 &= 0 \\ .6m_1 + .50m_2 - .66m_3 &= 0 \\ m_1 + m_2 + m_3 &= 1 \end{aligned}$$

$$m_3 = \frac{\begin{vmatrix} -.6 & .25 & 0 \\ .6 & .5 & 0 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} -.6 & .25 & .33 \\ .6 & .5 & -.66 \\ 1 & 1 & 1 \end{vmatrix}} = \frac{-.3 - .15}{-.267 - .711} = .460$$

$$m_1 = \frac{\begin{vmatrix} 0 & .25 & .33 \\ 0 & .5 & -.66 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} -.6 & .25 & .33 \\ .6 & .5 & -.66 \\ 1 & 1 & 1 \end{vmatrix}} = \frac{-.165 - .165}{-.978} = .337$$

$$.337 + m_2 + .460 = 1$$

$$m_2 = .203$$

a. (.3155, .2155, .469)

b. change of (.0019, -.0058, .0039)

c. (.337, .203, .460)

5. a. Travel agency "c"; steady state of 46%.

b. no

c. Implications: useful for corporation in forecasting management styles, customer service, etc.

6.

		To	
From	First	Regular	Economy
First	.80	.20	0
Regular	0	.40	.60
Economy	.10	.30	.60

$$(m_1,\ m_2,\ m_3) \begin{pmatrix} .8 & .2 & 0 \\ 0 & .4 & .6 \\ .1 & .3 & .6 \end{pmatrix} = (m_1,\ m_2,\ m_3)$$

$$.8m_1 + 0m_2 + .1m_3 = m_1$$

$$.2m_1 + .4m_2 + .3m_3 = m_2$$

$$.6m_2 + .6m_3 = m_3$$

$$
\begin{aligned}
-.2m_1 + 0m_2 + .1m_3 &= 0 \\
.6m_2 + .4m_3 &= 0 \\
m_1 + m_2 + m_3 &= 1
\end{aligned}
$$

$$
m_3 = \frac{\begin{vmatrix} -.2 & 0 & 0 \\ 0 & .6 & 0 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} -.2 & 0 & .1 \\ 0 & .6 & -.4 \\ 1 & 1 & 1 \end{vmatrix}} = \frac{-.12}{-.12 - .14} = .4615
$$

$$
\begin{aligned}
.6m_2 - .4(.4615) &= 0 \\
m_2 &= .3077 \\
m_1 + .3077 + 4615 &= 1 \\
m_1 &- .2308
\end{aligned}
$$

a. (.2308, .3077, .4615)

b. Implications: economy steady state of 46.15%.

CHAPTER 4

LINEAR PROGRAMMING CONCEPTS

GRAPHING MODELS

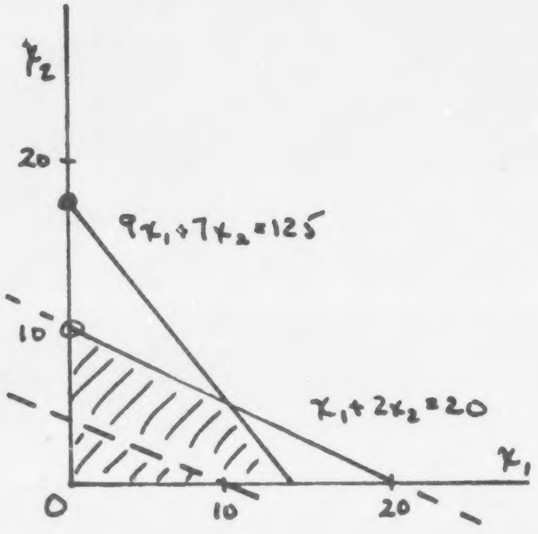
Linear programming concepts are discussed in this chapter. Terminology is introduced by the examples along with a step by step approach to solving various problems.

In addition to testing the basic points, the concept of the family of lines, or iso-profit and iso-cost functions, are demonstrated. Of special notes is the format introduced in example 2.

Word problems are presented to force the student to begin to realize that linear programming does have "real world" applications. Problems 16 and 17 are for "set up only."

Exercise 4.2, pp. 212-215.

1.



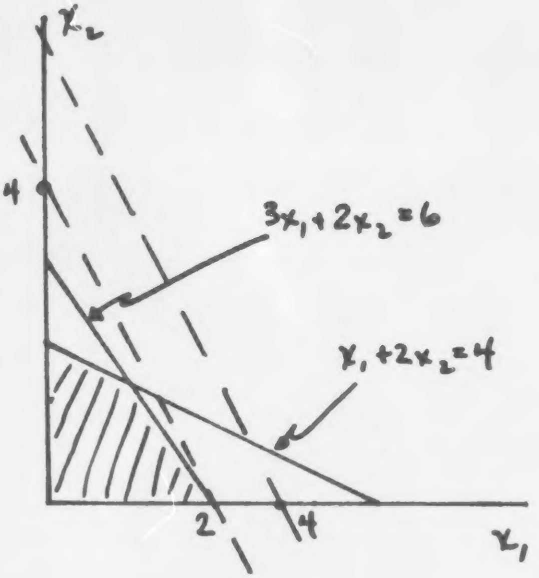
$$Z_{max} = 5x_1 + 10x_2$$

$$|A| = \begin{vmatrix} 9 & 7 \\ 1 & 2 \end{vmatrix} = 11$$

$$\bar{x}_1 = \frac{\begin{vmatrix} 125 & 7 \\ 20 & 2 \end{vmatrix}}{11} = 10$$

$z = 100 \qquad x_2 = 5$

2.



x_2
4
$3x_1+2x_2=6$
$x_1+2x_2=4$
2
4
x_1

$z_{max} = 6x_1 + 3x_2$

$x_1 = 2$

$x_2 = 0$

$z = 12$

3.

$z_{max} = 2x_1 + 5x_2$

$$|A| = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5$$

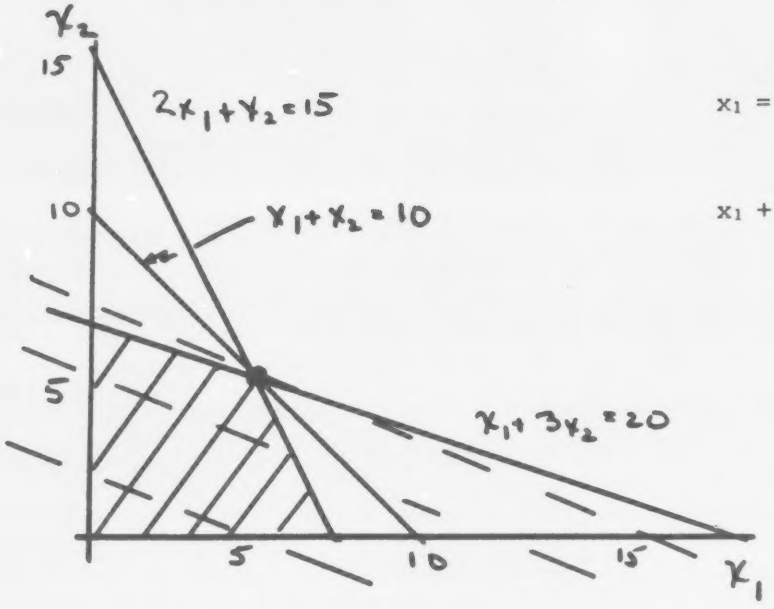
$$x_1 = \frac{\begin{vmatrix} 15 & 1 \\ 20 & 3 \end{vmatrix}}{5} = 5$$

$x_1 + 3x_2 = 20$

$x_1 = 5$

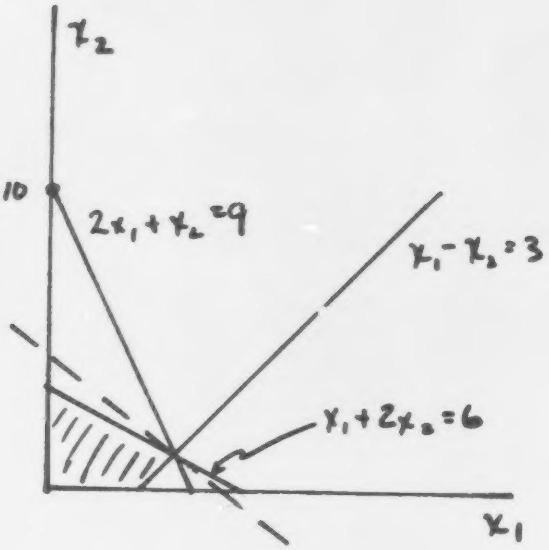
$x_2 = 5$

$z = 35$



x_2
15
$2x_1+x_2=15$
10
$x_1+x_2=10$
5
$x_1+3x_2=20$
5
10
15
x_1

4.


x_2
10
$2x_1 + x_2 = 9$
$x_1 - x_2 = 3$
$x_1 + 2x_2 = 6$
x_1

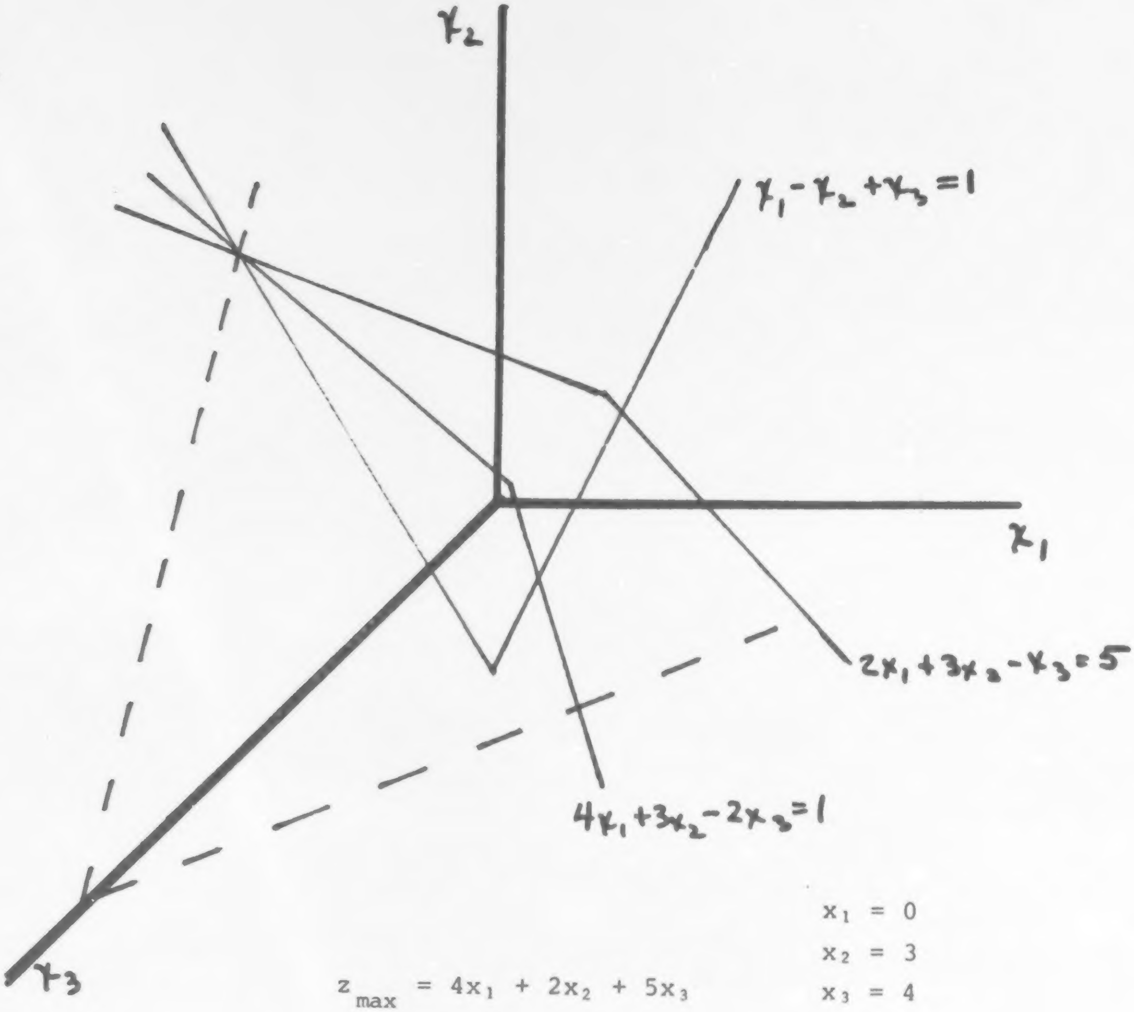
$z_{max} = 3x_1 + 4x_2$

$x_1 = 4$

$x_2 = 1$

$z = 16$

5.


x_2
$x_1 - x_2 + x_3 = 1$
x_1
$2x_1 + 3x_2 - x_3 = 5$
$4x_1 + 3x_2 - 2x_3 = 1$
x_3

$x_1 = 0$

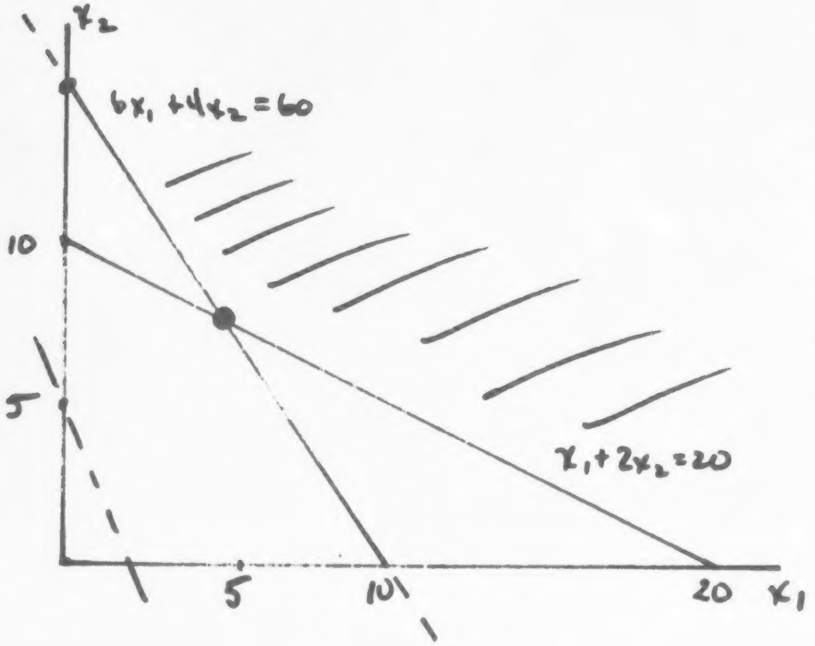
$x_2 = 3$

$x_3 = 4$

$z = 26$

$z_{max} = 4x_1 + 2x_2 + 5x_3$

6.


x_2
$6x_1+4x_2=60$
10
5
$x_1+2x_2=20$
5
10
20
x_1

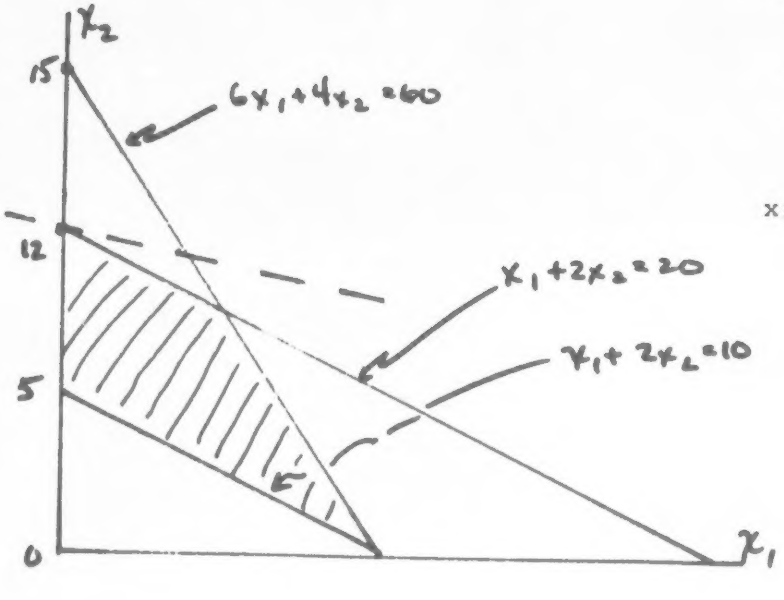
$$|A| = \begin{vmatrix} 6 & 4 \\ 1 & 2 \end{vmatrix} = 8$$

$$x_1 = \frac{\begin{vmatrix} 60 & 4 \\ 20 & 2 \end{vmatrix}}{8} = 5$$

$x_2 = 7.5$

$z_{\min} = 30$

7.


x_2
15
$6x_1+4x_2=60$
12
$x_1+2x_2=20$
$x_1+2x_2=10$
5
0
x_1

$z_{\max} = x_1 + 5x_2$

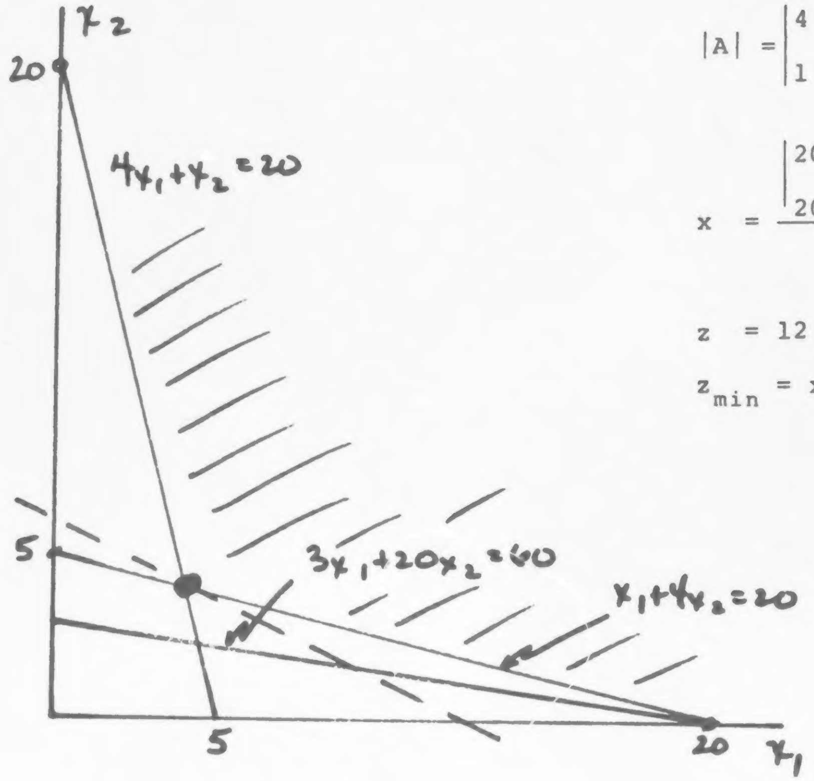
$x_1 = 0$

$x_2 = 10$

$z = 50$

$x_1 + 2x_2 = 20$

8.



$$|A| = \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = 15$$

$$x = \frac{\begin{vmatrix} 20 & 1 \\ 20 & 4 \end{vmatrix}}{15} = 4$$

$z = 12 \quad x_2 = 4$

$z_{min} = x_1 + 2x_2$

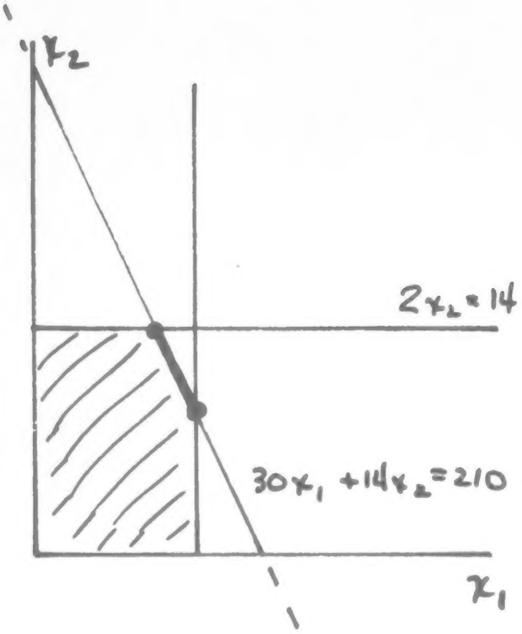
9.

$z_{max} = 15x_1 + 7x_2$

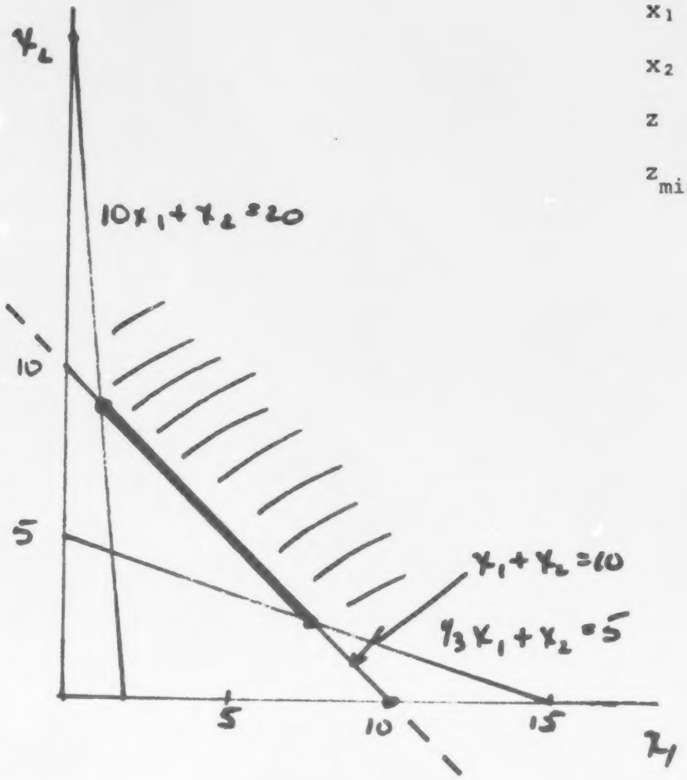
$x_1 = 5,\ 3.730$

$x_2 = 4.2860,\ 7$

$z = 105.002$



10.



x_2
$10x_1 + x_2 = 20$
10
5
$x_1 + x_2 = 10$
$1/3\, x_1 + x_2 = 5$
5
10
15
x_1

$x_1 = 1.111,\ 15/2$

$x_2 = 8.888+,\ 5/2$

$z = 20$

$z_{min} = 2x_1 + 2x_2$

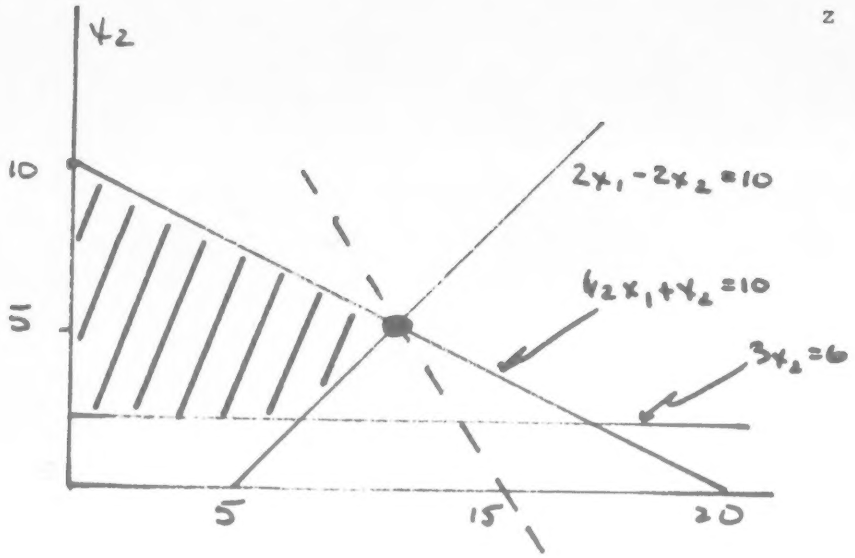
11.

$z_{max} = 3x_1 + 2x_2$

$x_1 = 10$

$x_2 = 5$

$z = 40$



x_2
10
5
$2x_1 - 2x_2 = 10$
$1/2\, x_1 + x_2 = 10$
$3x_2 = 6$
5
15
20

12.

$$z_{max} = 9x_1 + 21x_2$$

subject to:

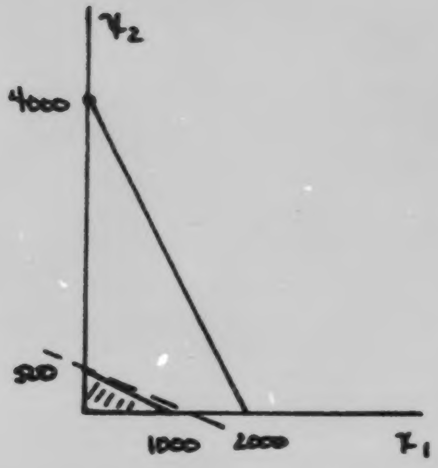
$$10x_1 + 20x_2 \le 10{,}000$$

$$4x_1 + 2x_2 \le 8{,}000$$

$$x_i \ge 0$$

x_1 = box

x_2 = drum



13.

$$z_{max} = 10x_1 + 20x_2 + 5x_3$$

subject to:

$$x_1 \ge 100$$

$$x_2 \ge 25$$

$$x_3 \le 10$$

$$x_1 + 10x_2 + 12x_3 \le 2000$$

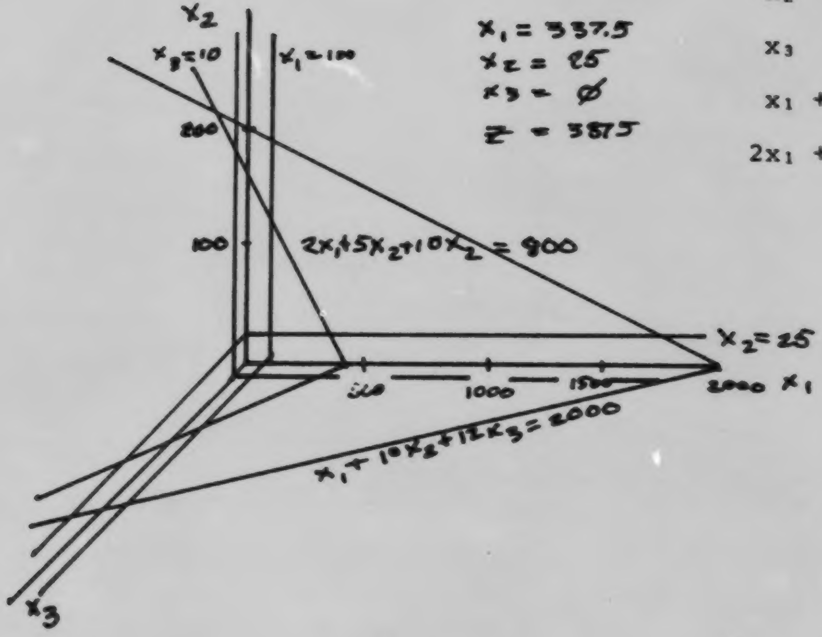
$$2x_1 + 5x_2 + 10x_3 \le 800$$

$$x_i \ge 0$$

x_1 = chairs

x_2 = desks

x_3 = bookcases



$x_1 = 337.5$

$x_2 = 25$

$x_3 = 0$

$z = 3875$

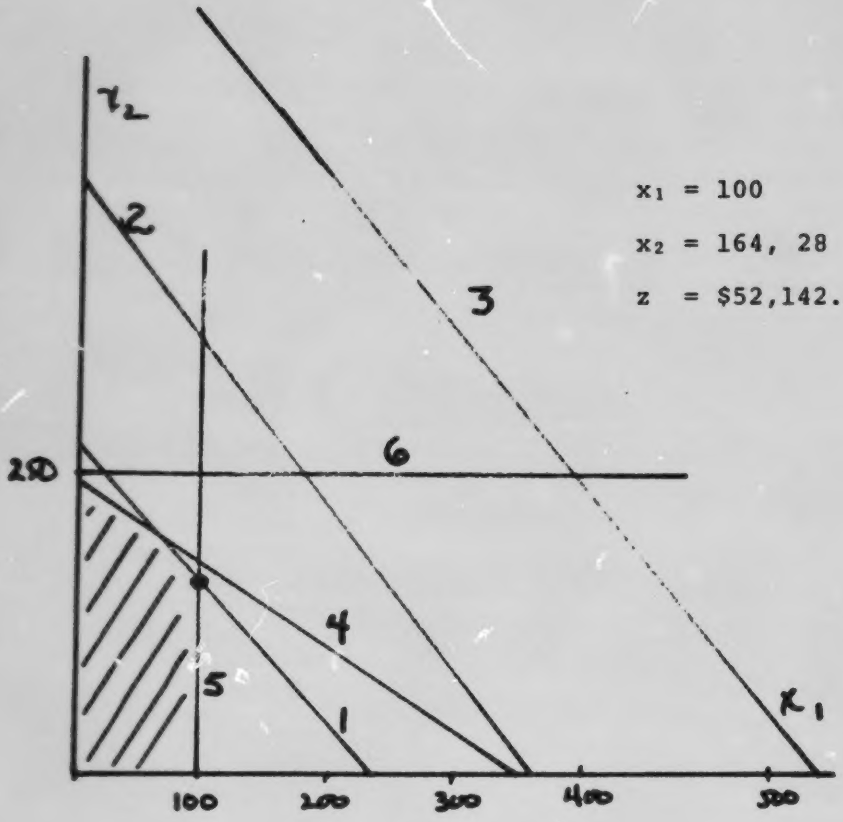
14. x_1 = large table

x_2 = small table

$z_{max} = 275x_1 + 150x_2$

subject to:

$$
\begin{aligned}
17x_1 + 14x_2 &\le 4000 && (1)\\
14x_1 + 10x_2 &\le 5000 && (2)\\
15x_1 + 8x_2 &\le 8000 && (3)\\
20x_1 + 26x_2 &\le 7000 && (4)\\
x_1 &\le 100 && (5)\\
x_2 &\le 250 && (6)\\
x_i &\ge 0 && (i = 1, 2)
\end{aligned}
$$



$x_1 = 100$

$x_2 = 164, 28$

$z = \$52,142.86$

15. x_1 = torpedo

x_2 = submarine

$z_{max} = \$1.25x + 1.50x$

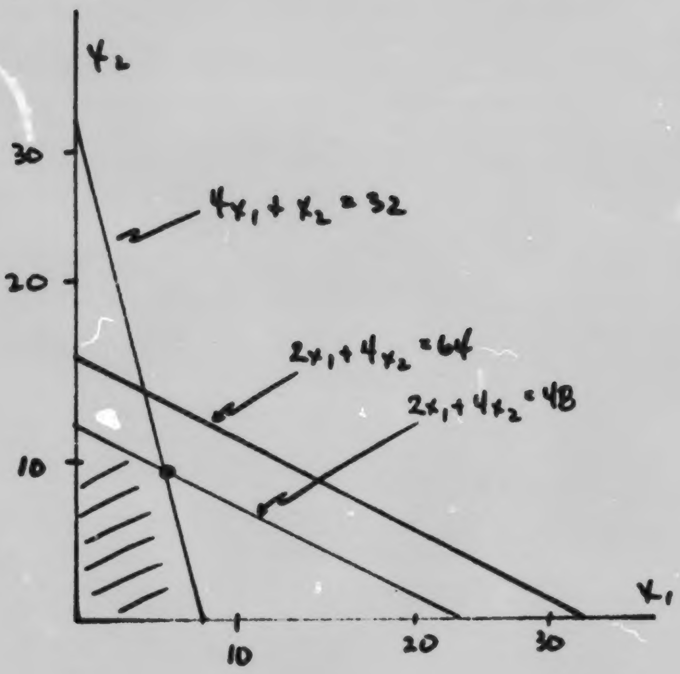
subject to:

$$2x_1 + 4x_2 \leq 48 \qquad (1)$$

$$4x_1 + 1x_2 \leq 32 \qquad (2)$$

$$2x_1 + 4x_2 \leq 64 \qquad (3)$$

$$x_i \geq 0 \qquad (i = 1, 2)$$



$x_1 = 5.71$

$x_2 = 9.14$

$z_{max} = \$20.85$

16.

Ingredients	Wine Punch	Nonalcoholic Punch
Burgundy	x_1	
Chablis	x_2	
Sangria	x_3	
7-Up	x_4	x_5
Apple juice		x_5
Ginger Ale		x_7

$$z_{min} = .05x_1 + .06x_2 + .03x_3 \quad .01x_4 + .01x_5 + .04x_6 + .015x_7$$

subject to:

$$x_1 + x_2 + x_3 + x_4 = 1000$$

$$x_1 \le 320$$

$$* \left(\begin{array}{l} x_1 \ge 100 \\ x_1 \ge 0.10(x_1 + x_2 + x_3 + x_4) \end{array} \right)$$

$$x_2 \le 128$$

$$x_3 \le 320$$

$$* \left(\begin{array}{l} x_3 \ge 200 \\ x_3 \ge 0.20(x_1 + x_2 + x_3 + x_4) \end{array} \right)$$

$$x_5 + x_6 + x_7 = 200$$

$$x_4 + x_5 \le 320$$

$$* \left(\begin{array}{l} x_4 \le 250 \\ x_4 \le 0.25(x_1 + x_2 + x_3 + x_4) \end{array} \right)$$

$$x_6 \le 64$$

$$* \left(\begin{array}{c} 20 \le x_6 \le 40 \\ 0.10(x_5 + x_6 + x_7) \le x_6 \le 0.20(x_5 + x_6 + x_7) \end{array} \right)$$

$$x_i \ge 0 \ (i = 1, 2, 3, 4, 5, 6, 7)$$

* = redundant equations

The above can be shown as:

$$z_{min} = .05x_1 + .06x_2 + .03x_3 + .01x_4 + .01x_5 + .04x_6 + .015x_7$$

subject to:

$$x_1 + x_2 + x_3 + x_4 = 1000$$

$$x_5 + x_6 + x_7 = 200$$

$$x_1 \ge 100$$

$$x_3 \ge 200$$

$$x_4 \le 250$$

$$x_6 \le 40$$

$$x_6 \ge 20$$

$$x_1 \le 320$$

$$x_2 \le 128$$

$$x_3 \le 320$$

$$x_4 + x_5 \le 320$$

$$x_6 \le 64$$

$$x_i \ge 0$$

17. x_1 = number of pieces of mast made on lathe A, deburrer C

x_2 = number of pieces of mast made on lathe A, deburrer D

x_3 = number of pieces of mast made on lathe A, deburrer E

x_4 = number of pieces of mast made on lathe B, deburree C

x_5 = number of pieces of mast made on lathe B, deburrer D

x_6 = number of pieces of mast made on lathe B, deburrer E

x_7 = number of pieces of bushing made on lathe A, deburrer C

x_8 = number of pieces of bushing made on lathe B, deburrer C

x_9 = number of pieces of brackets made on lathe B, deburrer D

objective function:

for x_1 costs 1.5(9.00) = \$13.50 lathe A

0.6(4.50) = 2.70 deburrer C

raw materials = 1.10

\$17.30

revenue \$29.60

profit = revenue - costs = \$29.60 - 17.30 = \$12.30

$x_2 = (29.60 - 1.10) - (13.50 + 2.25) = \$12.75$

$x_3 = 28.50 - (13.50 + 1.46) = 13.54$

$x_4 = 28.50 - (7.32 + 2.70) = 18.48$

$x_5 = 28.50 - (7.32 + 2.25) = 18.93$

$x_6 = 28.50 - (7.32 + 1.46) = 19.72$

$x_7 = 18.70 - (11.70 + 2.70) = 4.30$

$x_8 = 18.70 - (5.07 + 2.70) = 10.93$

$x_9 = 17.82 - (6.19 + 2.50) = 9.13$

Summary Cost Table

	Mast	Bushing	Bracket
A	13.50	11.70	-----
B	7.32	5.07	6.19
C	2.70	2.70	-----
D	2.25	-----	2.50
E	1.46	-----	-----
	28.50	18.70	17.32

$$z_{max} = 12.30x_1 + 12.75x_2 + 13.54x_3 + 18.48x_4 + 18.93x_5 + 19.72x_6 + 4.30x_7 + 10.93x_8 + 9.13x_9$$

subject to:

$$\begin{aligned}
1.5x_1 + 1.5x_2 + 1.5x_3 + 1.3x_7 &\le 90 \\
1.3x_4 + 1.3x_5 + 1.3x_6 + 0.9x_8 + 1.1x_9 &\le 75 \\
0.6x_1 + 0.6x_4 + 0.6x_7 + 0.6x_8 &\le 60 \\
.45x_2 + .45x_5 + 0.5x_9 &\le 100 \\
0.3x_3 + 0.3x_6 &\le 65 \\
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 &\ge 40 \\
x_7 + x_8 &\ge 45 \\
x_9 &\ge 30
\end{aligned}$$

$x_i \ge 0$ (1, 2, 3, 4, 5, 6, 7, 8, 9)

CHAPTER 5

LINEAR PROGRAMMING CONCEPTS

To capture the students' interest, the "situation" involving the Jones Services Company and General Forms results in a merger. The merger results in $2016 more profit than the joint profits of the two companies.

In presenting the simplex algorithm, the tableau follows a graphic explantion. The algorithm is presented mechanically and with economic interpretations.

Steps for solving a linear programming problem are summarized on page 241. When I present the concept of leaving variable, I describe the selection by "the smallest positive ratio." Technically, zero can also be used. This should be mentioned in class--even though your explanation of production trade-off dissolves when you leave zero resources (which provides the b_i/a_{ij}).

Problem 10 on page 243 is a lead in for minimizing problems. It can be solved by graphing before discussing section 5.4.

When presenting the Big-M method, I stress that when attempting to solve by the simplex algorithm, one needs to take into account whether a constraint is (1) equal to, (2) equal to or less than, or (3) equal to or greater than. Situations one and three require an artificial variable whether the objective function is to be maximized or minimized.

Duality is shown by a "wiring diagram" on page 265 and by relating initial and final tableaux on pages 270-271.

The problems have been worked to completion and are presented in the event you wish to photocopy specific problems for class distribution.

Sensitivity analysis is presented in the Study Guide.

Exercise 5.2, pp. 241-244.

1.

C_j		12	10	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	b
0	s_1	3	2	1	0	6
0	s_2	1	2	0	1	4
	Z_j	0	0	0	0	0
	$C_j - Z_j$	12	10	0	0	

C_j		12	10	0	0	
	Sol. Mix	x_1	x_2	s_1	s_1	b
12	x_1	1	2/3	1/3	0	2
0	s_2	0	+4/3	-1/3	+1	+2
	Z_j	12	8	4	0	24
	$C_j - Z_j$	0	2	-4	0	

C_j		12	10	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	b
12	x_1	1	0	1/3	-1/2	1
10	x_2	0	1	-1/4	3/4	3/2
	Z_j	12	10	3/2	3/2	27
	$C_j - Z_j$	0	0	-3/2	-3/2	

2.

C_j		2	5	0	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	s_3	b
0	s_1	1	1	1	0	0	10
0	s_2	2	1	0	1	0	15
0	s_3	1	3	0	0	1	20
	Z_j	0	0	0	0	0	0
	$C_j - Z_j$	2	5	0	0	0	

C_j		2	5	0	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	s_3	b
0	s_1	2/3	0	1	0	-1/3	10/3
0	s_2	5/3	0	0	1	-1/3	25/3
5	x_2	1/3	1	0	0	1/3	20/3
	Z_j	5/3	5	0	0	5/3	100/3
	$C_j - Z_j$	1/3	0	0	0	-5/3	

C_j		2	5	0	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	s_3	b
z	x_1	1	0	3/2	0	-1/2	5
0	s_2	0	0	-5/2	1	1/2	0
5	x_2	0	1	-1/2	0	1/2	5
	Z_j	2	5	1/2	0	3/2	35
	$C_j - Z_j$	0	0	-1/2	0	-3/2	

3.

C_j		3	4	0	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	s_3	b
0	s_1	2	1	1	0	0	9
0	s_2	1	-1	0	1	0	3
0	s_3	1	2	0	0	1	6
	Z_j	0	0	0	0	0	0
	$C_j - Z_j$	3	4	0	0	0	

C_j		3	4	0	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	s_3	b
0	s_1	3/2	0	1	0	-1/2	6
0	s_2	3/2	0	0	1	3/2	6
4	x_2	1/2	1	0	0	1/2	3
	Z_j	2	4	0	0	2	12
	$C_j - Z_j$	1	0	0	0	-2	

C_j		3	4	0	0	0	
	Sol. Mix	x	x	s	s	s	b
0	s_1	0	0	1	-1	-2	0
3	x_1	1	0	0	2/3	1	4
4	x_2	0	1	0	-1/3	0	1
	Z_j	3	4	0	2/3	3	16
	$C_j - Z_j$	0	0	0	-2/3	-3	

4.

C_j		15	7	0	0	0	
	Sol. Mix	x	x	s	s	s	b
0	s_1	3	0	1	0	0	15
0	s_2	0	2	0	1	0	14
0	s_3	30	14	0	0	1	210
	Z_j	0	0	0	0	0	0
	$C_j - Z_j$	15	7	0	0	0	

C_j		15	7	0	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	s_3	b
15	x_1	1	0	1/3	0	0	5
0	s_2	0	2	0	1	0	14
0	s_3	0	14	-10	1	0	60
	Z_j	15	0	5	0	0	75
	$C_j - Z_j$	0	7	-5	0	0	

C_j		15	7	0	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	s_3	b
15	x_1	1	0	1/3	0	0	5
0	s_2	0	0	10/7	6/7	0	38/7
7	x_2	0	1	-10/14	1/14	0	60/14
	Z_j	15	7	0	1/2	0	105
	$C_j - Z_j$	0	0	0	-1/2	0	

5.

C_j		5	4	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	b
0	s_1	3	1	1	0	60
0	s_2	1	2	0	1	60
	Z_j	0	0	0	0	0
	$C_j - Z_j$	5	4	0	0	

C_j		5	4	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	b
5	x_1	1	1/3	1/3	0	20
0	s_2	0	5/3	-1/3	1	40
	Z_j	5	5/3	5/3	0	100
	$C_j - Z_j$	0	7/3	-5/3	0	

C_j		5	4	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	b
5	x_1	1	0	6/15	-1/5	12
4	x_2	0	1	-1/5	3/5	24
	Z_j	5	4	6/5	7/5	156
	$C_j - Z_j$	0	0	-6/5	-7/5	

6.

C_j		50	30	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	b
0	s_1	6	3	1	0	900
0	s_2	3	4	0	1	600
	Z_j	0	0	0	0	0
	$C_j - Z_j$	50	30	0	0	

C_j		50	30	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	b
50	x_1	1	1/2	1/6	0	150
0	s_2	0	5/2	-1/2	1	150
	Z_j	50	25	25/3	0	7500
	$C_j - Z_j$	0	5	-25/3	0	

C_j		50	30	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	b
50	x_1	1	0	8/30	-1/5	120
30	x_2	0	1	-1/5	2/5	60
	Z_j	50	30	22/3	2	7500
	$C_j - Z_j$	0	0	-22/3	-2	

7.

C_j		9	21	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	b
0	s_1	4	2	1	0	8000
0	s_2	10	20	0	1	10,000
	Z_j	0	0	0	0	0
	$C_j - Z_j$	9	21	0	0	

C_j		9	21	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	b
0	s_1	3	0	1	-1/10	7000
21	x_2	1/2	1	0	1/20	500
	Z_j	21/2	21	0	21/20	10,500
	$C_j - Z_j$	-3/2	0	0	-21/20	

8. x_1 = peanut brittle

x_2 = cashew brittle

x_3 = pistachio brittle

cost of nuts per pound of brittle:

$x_1 = \$1.00(1/2) = 1/2$

$x_2 = \$2.00(1/4) = 1/2$

$x_3 = \$4.00(1/2) = 2$

profit per type of brittle:

$x_1 = 2 - 1/2 = 1\ 1/2$

$x_2 = 6 - 1/2 = 5\ 1/2$

$x_3 = 8 - 2 = 6$

$Z_{max} = 1.5x_1 + 5.5x_2 + 6x_3$

subject to:

$$x_1 + x_2 + x_3 \le 16$$

$$1/2x_1 + 1/2x_2 + 2x_3 \le 10$$

$$x_3 \le 2$$

C_j			1.5	5.5	6	0	0	0
	Sol. Mix	x_1	x_2	x_3	s_1	s_2	s_3	b
0	s_1	1	1	1	1	0	0	16
0	s_2	1/2	1/2	2	0	1	0	10
0	s_3	0	0	1	0	0	1	2
	Z_j	0	0	0	0	0	0	0
	$C_j - Z_j$	1.5	5.5	6	0	0	0	

C_j		1.5	5.5	6	0	0	0	
	Sol. mix	x_1	x_2	x_3	s_1	s_2	s_3	b
0	s_1	1	1	0	1	0	-1	14
0	s_2	1/2	1/2	0	0	1	-2	6
6	x_3	0	0	1	0	0	1	2
	Z_j	0	0	6	0	0	6	12
	$C_j - Z_j$	1.5	5.5	0	0	0	-6	

C_j		1.5	5.5	6	0	0	0	
	Sol. mix	x_1	x_2	x_3	s_1	s_2	s_3	b
0	s_1	0	0	0	1	-2	3	2
5.5	x_2	1	1	0	0	2	-4	12
6	x_3	0	0	1	0	0	1	2
	Z_j	5.5	5.5	6	0	11	-16	78
	$C_j - Z_j$	-4	0	0	0	-11	16	

C_j		1.5	5.5	6	0	0	0	
	Sol mix	x_1	x_2	x_3	s_1	s_2	s_3	b
0	s_3	0	0	0	1/3	-2/3	1	2/3
5.5	x_2	1	1	0	4/3	-2/3	0	44/3
6	x_3	0	0	1	-1/3	2/3	0	4/3
	Z_j	5.5	5.5	6	16/3	1/3	0	88 2/3
	$C_j - Z_j$	-4	0	0	-16/3	-1/3	0	

Peanut brittle = 0 lbs

Cashew brittle = 14 2/3 lbs

Pistachio brittle = 1 1/3 lbs

9. x_1 = chocolate chip

x_2 = fudge chocolate chip

$$Z_{max} = 1.20x_1 + 1.30x_2$$

subject to:

$$\begin{aligned} x_1 + \quad x_2 &\le 400 \\ 0.02x_1 + 0.04x_2 &\le 10 \\ x_2 &\le 120 \\ x_i &\ge 0 \quad (i = 1, 2) \end{aligned}$$

C_j		1.2	1.3	0	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	s_3	b
0	s_1	1	1	1	0	0	400
0	s_2	.02	.04	0	1	0	10
0	s_3	0	1	0	0	1	120
	Z_j	0	0	0	0	0	0
	$C_j - Z_j$	1.2	1.3	0	0	0	

C_j		1.2	1.3	0	0	0	
	Sol. mix	x_1	x_2	s_1	s_2	s_3	b
0	s_1	1	0	1	0	-1	280
0	s_2	.02	0	0	1	-.04	5.2
1.3	x_2	0	1	0	0	1	120
	Z_j	0	1.3	0	0	1.3	156
	$C_j - Z_j$	1.2	0	0	0	-1.3	

C_j		1.2	1.3	0	0	0	
	Sol. mix	x_1	x_2	s_1	s_2	s_2	b
0	s_1	0	0	1	-50	1	20
1.2	x_1	1	0	0	50	-2	260
1.3	x_2	0	1	0	0	1	120
	Z_j	1.2	1.3	0	60	-1.1	468
	$C_j - Z_j$	0	0	0	-60	1.1	

C_j		1.2	1.2	0	0	0	
	Sol. mix	x_1	x_2	s_1	s_2	s_3	b
0	s_3	0	0	1	-50	1	20
1.2	x_1	1	0	2	-50	0	300
1.3	x_2	0	1	-1	50	0	100
	Z_j	1.2	1.3	1.1	5	0	490
	$C_j - Z_j$	0	0	-1.1	-5	0	

10. Caution: This problem is beyond students' ability. It sets the stage for the Big-M method discussed in section 5.4 and 5.5. (Students can solve it graphically.)

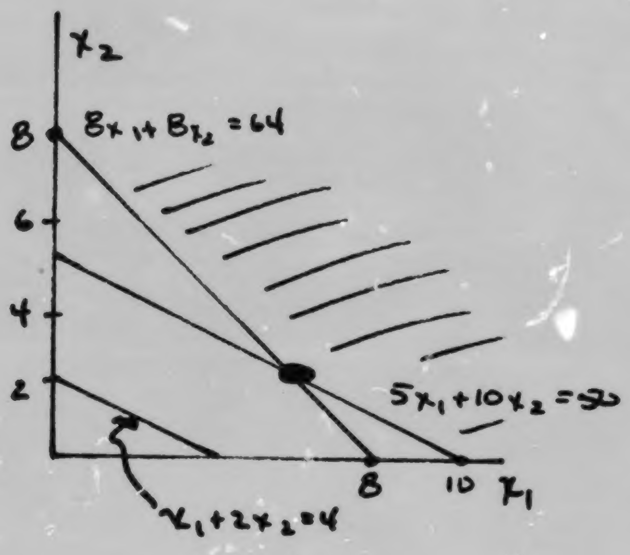
x_1 = Boise Cascade

x_2 = San Diego Office Supply

$Z_{min} = 37x_1 + 49x_2$

subject to:

$$
\begin{aligned}
x_1 + 2x_2 &\geq 4 \\
8x_1 + 8x_2 &\geq 64 \\
5x_1 + 10x_2 &\geq 50 \\
x_i &\geq 0
\end{aligned}
$$



C_j		37	49	0	0	0	M	M	M	
	Sol. Mix	x_1	x_2	s_1	s_2	s_3	A_1	A_2	A_3	b
M	A_1	1	2	-1	0	0	1	0	0	4
M	A_2	8	8	0	-1	0	0	1	0	64
M	A_3	5	10	0	0	-1	0	0	1	50
	Z_j	14M	20M	-M	-M	-M	M	M	M	118M
	$C_j - Z_j$	37-19M	49-20M	M	M	M	0	0	0	

11. x_1 = number of regular sandcasting - lathe 1

x_2 = number of regular sandcasting - lathe 2

x_3 = number of four-barrel sandcasting - lathe 1

x_4 = number of fuel injection sandcasting - lathe 2

$Z = 12.50x_1 + 12.50x_2 + 17.50x_3 + 22.50x_4$

subject to:

$$100X_1 + 100x_2 + 150x_3 + 175x_4 \le 40{,}000 \text{ (sand)}$$

$$15x_1 + 27x_3 \le 10{,}000 \text{ (lathe 1)}$$

$$14x_2 + 45x_4 \le 9{,}000 \text{ (lathe 2)}$$

$$x_3 \le 100$$

$$x_4 \le 500$$

$$-x_1 - x_2 + 2x_3 + 2x_4 \le 0$$

$$x_i \ge 0$$

C_j		12.50	12.50	17.50	22.50	0	0	0	0	0	0	
	Sol. mix	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	s_5	s_6	b
0	s_1	100	100	150	175	1	0	0	0	0	0	40,000
0	s_2	15	0	27	0	0	1	0	0	0	0	10,000
0	s_3	0	14	0	45	0	0	1	0	0	0	9,000
0	s_4	0	0	1	0	0	0	0	1	0	0	100
0	s_5	0	0	0	1	0	0	0	0	1	0	500
0	s_6	-1	-1	2	2	0	0	0	0	0	1	0
	Z_j	0	0	0	0	0	0	0	0	0	0	0
	$C_j - Z_j$	12.50	12.50	17.50	22.50	0	0	0	0	0	0	

C_j		12.5	12.5	17.5	22.5	0	0	0	0	0	0	
	Sol. Mix	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	s_5	s_6	b
0	s_1	375/2	375/2	-25	0	1	0	0	0	0	-175/2	40,000
0	s_2	15	0	27	0	0	1	0	0	0	0	10,000
0	s_3	45/2	73/2	-45	0	0	0	1	0	0	-45/2	9,000
0	s_4	0	0	1	0	0	0	0	1	0	0	100
0	s_5	1/2	1/2	-1	0	0	0	0	0	1	-1/2	500
22.5	x_4	-1/2	-1/2	1	1	0	0	0	0	0	1/2	0
	Z_j	-45/4	-45/4	22.5	22.5	0	0	0	0	0	45/4	0
	$C_j - Z_j$	5/4	5/4	-5.5	0	0	0	0	0	0	-45/4	

C_j		12.5	12.5	17.5	22.5	0	0	0	0	0	0	
	Sol. mix	x	x	x	x	s	s	s	s	s	s	b
12.5	x_2	1	1	-2/15	0	2/375	0	0	0	0	-7/15	213.33
0	s_2	15	0	27	0	0	1	0	0	0	0	10,000
0	s_3	-14	0	73/15	0	-73/375	0	1	0	0	511/30	1213.33
0	s_4	0	0	1	0	0	0	0	1	0	0	100
0	s_5	0	0	-14/15	0	-1/375	0	0	0	1	-8/30	393.33
22.5	x_4	0	0	14/15	1	1/375	0	0	0	0	8/30	106.67
	Z_j	12.5	12.5	58/3	22.5	19/150	0	0	0	0	1/36	5066.67
	$C_j - Z_j$	0	0	-11/6	0	-19/150	0	0	0	0	-1/36	

12. [Constraint 4 must be multiplied by (-1).]

Cj		.0288	.0224	0	0	0	0	0	0	
	Sol. mix	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6	b
0	s_1	9/8	7/8	1	0	0	0	0	0	4800
0	s_2	1	0	0	1	0	0	0	0	3500
0	s_3	0	1	0	0	1	0	0	0	3500
0	s_4	-1	3/2	0	0	0	1	0	0	0
0	s_5	1	1	0	0	0	0	1	0	5000
0	s_6	13/4	5/2	0	0	0	0	0	1	153,600
	Z_j	0	0	0	0	0	0	0	0	0
	$C_j - Z_j$	.0288	.0224	0	0	0	0	0		

C_j		.0288	.0224	0	0	0	0	0	0	
	Sol. mix	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6	b
0	s_1	0	7/8	1	-9/8	0	0	0	0	862.50
.0288	x_1	1	0	0	1	0	0	0	0	3500
0	s_3	0	1	0	0	1	0	0	0	3500
0	s_4	0	3/2	0	1	0	1	0	0	3500
0	s_5	0	1	0	-1	0	0	1	0	1500
0	s_6	0	5/2	0	-13/4	0	0	0	1	142,225
	Z_j	.0288	0	0	.0288	0	0	0	0	100.80
	$C_j - Z_j$	0	.0224	0	-.0288	0	0	0	0	

C_j		.0288	.0224	0	0	0	0	0	0	
	Sol. mix	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6	b
.0224	x_2	0	1	8/7	-9/7	0	0	0	0	985.71
.0288	x_1	1	0	0	1	0	0	0	0	3500
0	s_3	0	0	-8/7	9/7	1	0	0	0	2514.29
0	s_4	0	0	-12/7	41/14	0	1	0	0	2021.44
0	s_5	0	0	-8/7	2/7	0	0	1	0	514.29
0	s_6	0	0	-20/7	-1/28	0	0	0	1	139,760.73
	Z_j	.0288	.0224	8/7	0	0	0	0	0	122.88
	$C_j - Z_j$	0	0	-8/7	0	0	0	0	0	

Non-binding constraints: numbers 3, 4, 5, 6.

Exercise 5.4, pp. 257-258.

1.

C_j		1	5	0	0	-M	0	
	Sol. Mix	x_1	x_2	s_1	s_2	A_1	s_3	b
0	s_1	1	2	1	0	0	0	20
-M	A_1	1	2	0	-1	1	0	10
0	s_3	6	4	0	0	0	1	60
	Z_j	-M	-2M	0	M	-M	0	-10M
	$C_j - Z_j$	1+M	5+2M	0	-M	0	0	

C_j		1	5	0	0	-M	0	
	Sol. mix	x_1	x_2	s_1	s_2	A_1	s_3	b
0	s_1	0	0	1	1	-1	0	10
5	x_2	1/2	1	0	-1/2	1/2	0	5
0	s_3	4	0	0	2	-2	1	40
	Z_j	5/2	5	0	-5/2	5/2	0	25
	$C_j - Z_j$	-3/2	0	0	5/2	-M-5/2	0	

C_j		1	5	0	0	-M	0	
	Sol. mix	x_1	x_2	s_1	s_2	A_1	s_3	b
0	s_2	0	0	1	1	-1	0	10
5	x_2	1/2	1	1/2	0	0	0	10
0	s_3	4	0	-2	0	0	1	20
	Z_j	5/2	5	5/2	0	0	0	50
	$C_j - Z_j$	-3/2	0	-5/2	0	-M	0	

2.

C_j		3	2	0	0	-M	0	
	Sol. Mix	x_1	x_2	s_1	s_2	A_1	s_3	b
0	s_1	2	-2	1	0	0	0	10
-M	A_1	0	3	0	-1	1	0	6
0	s_3	1	2	0	0	0	1	20
	Z_j	0	-3M	0	M	-M	0	-6M
	$Cj - Zj$	3	2+3M	0	-M	0	0	

C_j		3	2	0	0	-M	0	
	Sol. Mix	x_1	x_2	s_1	s_2	A_1	s_3	b
0	s_1	2	0	1	-2/3	2/3	0	14
2	x_2	0	1	0	-1/3	1/3	0	2
0	s_3	1	0	0	2/3	-2/3	1	16
	Z_j	0	2	0	-2/3	2/3	0	4
	$C_j - Z_j$	3	0	0	2/3	-M-2/3	0	

C_j		3	2	0	0	-M	0	
	Sol. Mix	x_1	x_2	s_1	s_2	A_1	s_3	b
3	x_1	1	0	1/2	-1/3	1/3	0	7
2	x_2	0	1	0	-1/3	1/3	0	2
0	s_3	0	0	-1/2	1	-1	1	9
	Z_j	3	2	3/2	-5/3	5/3	0	25
	$C_j - Z_j$	0	0	-3/2	5/3	-M-5/3	0	

C_j		3	2	0	0	-M	0	
	Sol. mix	x_1	x_2	s_1	s_2	A_1	s_3	b
3	x_1	1	0	1/3	0	0	1/3	10
2	x_2	0	1	-1/6	0	0	1/3	5
0	s_2	0	0	-1/2	1	-1	1	9
	Z_j	3	2	2/3	0	0	5/3	40
	$C_j - Z_j$	0	0	-2/3	0	-M	-5/3	

3.

C_j		8	4	0	0	0	-M	0	-M	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	s_3	A_1	s_4	A_2	s_5	s_6	b
0	s_1	4	-5	1	0	0	0	0	0	0	0	28
0	s_2	1	2	0	1	0	0	0	0	0	0	10
-M	A_1	3	0	0	0	-1	1	0	0	0	0	6
-M	A_2	0	5	0	0	0	0	-1	1	0	0	15
0	s_5	0	1	0	0	0	0	0	0	1	0	10
0	s_6	4	2	0	0	0	0	0	0	0	1	40
	Z_j	-3M	-5M	0	0	M	-M	M	-M	0	0	-21M
	$C_j - Z_j$	8+3M	4+5M	0	0	-M	0	-M	0	0	0	

C_j		8	4	0	0	0	-M	0	-M	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	s_3	A_1	s_4	A_2	s_5	s_6	b
0	s_1	4	0	1	0	0	0	-1	1	0	0	35
0	s_2	1	0	0	1	0	0	2/5	2/5	0	0	4
-M	A_1	3	0	0	0	-1	1	0	0	0	0	6
4	x_2	0	1	0	0	0	0	-1/5	1/5	0	0	3
0	s_5	0	0	0	0	0	0	1/5	-1/5	1	0	7
0	s_6	4	0	0	0	0	0	2/5	-2/5	0	1	34
	Z_j	-3M	4	0	0	M	-M	-4/5	4/5	0	0	12-6M
	$C_j - Z_j$	8+3M	0	0	0	-M	0	4/5	-M-4/5	0	0	

C_j		8	4	0	0	0	-M	0	-M	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	s_3	A_1	s_4	A_2	s_5	s_6	b
0	s_1	0	0	1	0	4/3	-4/3	-1	1	0	0	27
0	s_2	0	0	0	1	1/3	-1/3	2/5	2/5	0	0	2
8	x_1	1	0	0	0	-1/3	1/3	0	0	0	0	2
4	x_2	0	1	0	0	0	0	-1/5	1/5	0	0	3
0	s_5	0	0	0	0	0	0	1/5	-1/5	1	0	7
0	s_6	0	0	0	0	4/3	-4/3	2/5	-2/5	0	1	26
	Z_j	8	4	0	0	-8/3	8/3	-4/5	4/5	0	0	28
	$C_j - Z_j$	0	0	0	0	8/3	-M-8/3	4/5	-M-4/5	0	0	

C_j		8	4	0	0	0	-M	0	-M	0	0	
	Sol. Mix	x_1	x_2	s_1	s_2	s_3	A_1	s_4	A_2	s_5	s_6	b
0	s_1	0	0	1	-4	0	0	-13/5	-3/5	0	0	19
0	s_3	0	0	0	3	1	-1	6/5	6/5	0	0	6
8	x_1	1	0	0	1	0	0	2/5	2/5	0	0	4
4	x_2	0	1	0	0	0	0	-1/5	1/5	0	0	3
0	s_5	0	0	0	0	0	0	1/5	-1/5	1	0	7
0	s_6	0	0	0	-4	0	0	-6/5	-2	0	1	18
	Z_j	8	4	0	8	0	0	12/5	4	0	0	44
	$C_j - Z_j$	0	0	0	-8	0	-7	-12/5	-M-4	0	0	

4. x_1 = wooden chairs

x_2 = desks

x_3 = bookcases

$Z_{max} = 10x_1 + 20x_2 + 5x_3$

subject to:

$$\begin{aligned} x_1 &\ge 100 \\ x_2 &\ge 25 \\ x_3 &\le 10 \\ x_1 + 10x_2 + 12x_3 &\le 2{,}000 \\ 2x_1 + 5x_2 + 10x_3 &\ge 800 \\ x_3 &\ge 0 \end{aligned}$$

C_j		10	20	5	0	0	0	0	0	$-M$	$-M$	
	Sol. Mix	x_1	x_2	x_3	s_1	s_2	s_3	s_4	s_5	A_1	A_2	b
$-M$	A_1	1	0	0	-1	0	0	0	0	1	0	100
$-M$	A_2	0	1	0	0	-1	0	0	0	0	1	25
0	s_3	0	0	1	0	0	1	0	0	0	0	10
0	s_4	1	10	12	0	0	0	1	0	0	0	2000
0	s_5	2	5	10	0	0	0	0	1	0	0	800
	Z_j	$-M$	$-M$	0	M	M	0	0	0	M	M	$-125M$
	$C_j - Z_j$	$10+M$	$20+M$	5	$-M$	$-M$	0	0	0	0	0	

C_j		10	20	5	0	0	0	0	0	-M	-M	
	Sol. mix	x_1	x_2	x_3	s_1	s_2	s_3	s_4	s_5	A_1	A_2	b
-M	A_1	1	0	0	-1	0	0	0	0	1	0	100
20	x_2	0	1	0	0	-1	0	0	0	0	1	25
0	s_3	0	0	1	0	0	1	0	0	0	0	10
0	s_4	1	0	12	0	10	0	1	0	0	-10	1750
0	s_5	2	0	10	0	5	0	0	1	0	-5	675
	Zj	-M	20	0	M	-20	0	0	0	-M	20	500-100M
	Cj - Zj	10+M	0	5	-M	20	0	0	0	0	-M-20	

C_j		10	20	5	0	0	0	0	0	-M	-M	
	Sol. mix	x_1	x_2	x_3	s_1	s_2	s_3	s_4	s_5	A_1	A_2	b
10	x_1	1	0	0	-1	0	0	0	0	1	0	100
20	x_2	0	1	0	0	-1	0	0	0	0	1	25
0	s_3	0	0	1	0	0	1	0	0	0	0	10
0	s_4	0	0	12	1	10	0	1	0	-1	-10	1650
0	s_5	0	0	10	2	5	0	0	1	-2	-5	475
	Z_j	10	20	0	-10	-20	0	0	0	10	20	1500
	$C_j - Z_j$	0	0	5	10	20	0	0	0	-M-10	-M-20	

Cj		10	20	5	0	0	0	0	0	-M	-M	
	Sol. mix	x_1	x_2	x_3	s_1	s_2	s_3	s_4	s_5	A_1	A_2	b
10	x_1	1	0	0	-1	0	0	0	0	1	0	100
20	x_2	0	1	2	2/5	0	0	0	1/5	-2/5	0	120
0	s_3	0	0	1	0	0	1	0	0	0	0	10
0	s_4	0	0	-8	-3	0	0	1	-2	3	0	700
0	s_2	0	0	2	2/5	1	0	0	1/5	-2/5	-1	95
	Zj	10	20	40	-8	0	0	0	4	2	0	3400
	Cj - Zj	0	0	-35	8	0	0	0	-4	-2	0	

C_j		10	20	5	0	0	0	0	0	_m	-M	
	Sol. mix	x_1	x_2	x_3	s_1	s_2	s_3	s_4	s_5	A_1	A_2	b
10	x_1	1	0	5	0	5/2	0	0	1/2	0	-5/2	337.5
20	x_2	0	1	0	0	-1	0	0	0	0	1	25
0	s_3	0	0	1	0	0	1	0	0	0	0	10
0	s_4	0	0	7	0	15/2	0	1	-1/2	0	-15/2	1412.5
0	s_1	0	0	5	1	5/2	0	0	1/2	-1	-5/2	237.5
	Z	10	20	50	0	5	0	0	5	0	-5	3875
	$C_j - Z_j$	0	0	-45	0	-5	0	0	-5	-M	-M-5	

5. x_1 = slow-growth, low risk stock number one

x_2 = slow-growth, low risk stock number two

x_3 = savings and loan

x_4= high-risk, fast growth stock

$Z_{max} = 1.k0x_1 + 1.12x_2 + 1.08x_3 + 1.80x_4$

subject to:

$$x_1 + x_2 + x_3 + x_4 = 10{,}000$$

$$* \left[\begin{array}{l} x_1 + x_2 \geq x_3 + x_4 \\ -x_1 - x_2 + x_3 + x_4 \leq 0 \end{array}\right] \qquad \text{*redundant}$$

$$x_3 \geq 2{,}000$$

$$x \geq 0 \ (i = 1, 2, 3)$$

C_j		1.10	1.12	1.08	1.80	0	0	0	-M	-M	
	Sol. mix	x_1	x_2	x_3	x_4	s_1	s_2	s_3	A_1	A_2	b
-M	A_1	1	1	1	1	0	0	0	1	0	10,000
0	s_2	-1	-1	1	1	0	1	0	0	0	0
-M	A_3	0	0	1	0	0	0	-1	0	1	2,000
	Z	-M	-M	-2M	-M	0	0	M	-M	-M	-12000M
	$C_j - Z_j$	*	**	***	****	0	0	-M	0	0	

*1.10 + M

**1.12 + M

***1.08 + 2M

****1.80 + M

C_j		1.10	1.12	1.08	1.80	0	0	0	-M	-M	
	Sol. mix	x_1	x_2	x_3	x_4	s_1	s_2	s_3	A_1	A_2	b
-M	A_1	2	2	0	0	0	-1	0	1	0	10,000
1.08	x_3	-1	-1	1	1	0	1	0	0	0	0
-M	A_2	1	1	0	-1	0	-1	-1	0	1	2,000
	Z_j	*	**	***	****	0	†	M	-M	-M	-12,000M
	$C_j - Z_j$	††	†††	0	.72-M	0	††††	-M	0	0	

*-1.08 - 3M

**-1.08 - 3M

*** 1.08

**** 1.08 + M

† 1.08 + 2M

†† 2.18 + 3M

††† 2.20 + 3M

††††-1.08 - 2M

C_j		1.10	1.12	1/08	1.80	0	0	0	-M	-M	
	Sol. mix	x_1	x_2	x_3	x_4	s_1	s_2	s_3	A_1	A_2	b
-M	A_1	0	0	0	2	0	1	2	1	-2	6,000
1.08	x_3	0	0	1	0	0	0	-1	0	1	2,000
1.12	x_2	1	1	0	-1	0	-1	-1	0	1	2.000
	Z_j	1.12	1.12	1.08	*	0	**	***	-M	****	*****
	$C_j - Z_j$	-.02	0	0	†	0	††	†††	0	††††	

C_j		1.10	1.12	1.08	1.80	0	0	0	-M	-M	
	Sol. mix	x_1	x_2	x_3	x_4	s_1	s_2	s_3	A_1	A_2	b
1.80	x_4	0	0	0	1	0	1/2	1	1/2	-1	3,000
1.08	x_3	0	0	1	0	0	0	-1	0	1	2,000
1.12	x_2	1	1	0	0	0	-1/2	0	1/2	0	5,000
	Z_j	1.12	1.12	1.08	1.80	0	.34	.72	1.46	-.72	13,160
	$C_j - Z_j$	-.02	0	0	0	0	-.34	-.72	*	**	

*-M - 1.46

**-M + .72

*-1.12 - 2M   † 2.92 + 2M
**-1.12 - M   †† 1.12 + M
***-2.20 - 2M   ††† 2.20 + 2M
**** 2.20 + 2M   ††††-2.20 - 3M
***** 4400- 6000M

Exercise 5.5, pp. 263-264.

1.

C_j		2	2	0	M	0	M	0	M	
	Sol. mix	x_1	x_2	s_1	A_1	s_2	A_2	s_3	A_3	b
M	A_1	10	1	-1	1	0	0	0	0	20
M	A_2	1	1	0	0	-1	1	0	0	10
M	A_3	1	3	0	0	0	0	-1	1	15
	Z_j	12M	5M	-M	M	-M	M	-M	M	45M
	$C_j - Z_j$	2-12M	2-5M	M	0	M	0	M	0	

C_j		2	2	0	M	0	M	0	M	
	Sol. mix	x_1	x_2	s_1	A_1	s_2	A_2	s_3	A_3	b
2	x_1	1	1/10	-1/10	1/10	0	0	0	0	2
M	A_2	0	9/10	1/10	-1/10	-1	1	0	0	8
M	A_3	0	29/10	1/10	-1/10	0	0	-1	1	13
	Z_j	2	a	b	c	-M	M	-M	M	4+21M
	$C_j - Z_j$	0	d	e	f	M	0	M	0	

a. 2/10 + 38/10M

b. -2/10 + 2/10M

c. 2/10 - 2/10M

d. 9/5 - 38/10M

e. 2/10 - 2/10M

f. -2/10 + 12/10M

C_j		2	2	0	M	0	M	0	M	
	Sol. mix	x_1	x_2	s_1	A_1	s_2	A_2	s_3	A_3	b
2	x_1	1	0	-3/29	3/29	0	0	1/29	-1/29	a
M	A_2	0	0	3/29	-2/29	-1	1	9/29	-9/29	b
2	x_2	0	1	1/29	-1/29	0	0	-10/29	10/29	c
	Z_j	2	2	d	e	-M	M	f	g	h
	$C_j - Z_j$	0	0	i	j	M	0	k	m	

a. 45/29

b. 115/29

c. 130/29

d. 8/29 + 2/29M

e. 4/29 - 2/29M

f. -18/29 + 9/29M

g. 18/29 - 9/29M

h. 350/29 + 115/29M

i. -8/29 - 2/29M

j. 31/29M - 4/29

k. 18/29 - 9/29M

m. 38/29M - 18/29

C_j		2	2	0	M	0	M	0	M	
	Sol. mix	x_1	x_2	s_1	A_1	s_2	A_2	s_3	A_3	b
2	x_1	1	0	-1/9	1/9	1/9	-1/9	0	0	10/9
0	s_3	0	0	2/9	-2/9	-29/9	29/9	1	-1	115/9
2	x_2	0	1	1/9	-1/9	-10/9	10/9	0	0	80/9
	Zj	2	2	0	0	-2	2	0	0	20
	$C_j - Z_j$	0	0	0	M	2	M-2	0	M	

2.

C_j		1	2	0	M	0	M	0	M	
	Sol. mix	x_1	x_2	s_1	A_1	s_2	A_2	s_3	A_3	b
M	A_1	4	1	-1	1	0	0	0	0	20
M	A_2	3	20	0	0	-1	1	0	0	60
M	A_3	1	4	0	0	0	0	-1	1	20
	Z_j	8M	25M	-M	M	-M	M	-M	M	100M
	$C_j - Z_j$	1-8M	2-25M	M	0	M	0	M	0	

C_j		1	2	0	M	0	M	0	M	
	Sol. mix	x_1	x_2	s_1	A_1	s_2	A_2	s_3	A_3	b
M	A_1	77/20	0	-1	1	1/20	-1/20	0	0	17
2	x_2	3/20	1	0	0	-1/20	1/20	0	0	3
M	A_3	2/5	0	0	0	1/5	-1/5	-1	1	8
	Zj	a	2	-M	M	b	c	-M	M	6+25M
	$C_j - Z_j$	d	0	M	0	e	f	M	0	

a. 3/10 + 85/20M

b. -1/10 + 1/4M

c. 1/10 - 1/4M

d. 7/10 - 85/20M

e. 1/10 - 1/4M

f. -1/10 + 1/4M

C_j		1	2	0	M	0	M	0	M	
	Sol. mix	x_1	x_2	s_1	A_1	s_2	A_2	s_3	A_3	b
1	x_1	1	0	-20/77	20/77	1/77	-1/77	0	0	340/77
2	x_2	0	1	3/77	-3/77	-4/77	4/77	0	0	180/77
M	A_3	0	0	8/77	-8/77	15/77	-5/77	-1	1	480/77
	Z_j	1	2	a	b	c	d	-M	M	e
	$C_j - Z_j$	0	0	f	g	h	i	M	0	

a. -14/77 + 8/77M
b. 14/77 - 8/77M
c. -3/77 + 15/77M
d. 3/77 - 15/77M
e. 100/11 + 480/77M
f. 14/77 - 8/77M
g. 85/77 - 14/77M
h. 3/77 - 15/77M
i. -3/77 + 92/77M

C_j		1	2	0	M	0	M	0	M	
	Sol. mix	x_1	x_2	s_1	A_1	s_2	A_2	s_3	A_3	b
1	x_1	1	0	-4/5	4/5	0	0	1/15	-1/15	4
2	x_2	0	1	1/15	-1/15	0	0	-4/15	4/15	4
0	s_2	0	0	8/15	-8/15	1	-1	-77/15	77/15	32
	Z_j	1	2	-2/3	2/3	0	0	-7/15	7/15	12
	$C_j - Z_j$	0	0	2/3	M-2/3	0	M	7/15	M-7/15	

3.

C_j		3	2	0	M	0	M	
	Sol. mix	x_1	x_2	s_1	A_1	s_2	A_2	b
M	A_1	1	2	-1	1	0	0	20
M	A_2	6	4	0	0	-1	1	60
	Z_j	7M	6M	-M	M	-M	M	80M
	$C_j - Z_j$	3-7M	2-6M	M	0	M	0	

C_j		3	2	0	M	0	M	
	Sol. mix	x_1	x_2	s_1	A_1	s_2	A_2	b
M	A_1	0	4/3	-1	1	1/6	-1/6	10
3	x_1	1	2/3	0	0	-1/6	1/6	10
	Z_j	3	2+4/3M	-M	M	-1/2+1/6M	1/2-1/6M	30+10M
	$C_j - Z_j$	0	-4/3M	M	0	1/2-1/6M	7/6M -1/2	

C_j		3	2	0	M	0	M	
	Sol. mix	x_1	x_2	s_1	A_1	s_2	A_2	b
2	x_2	0	1	-3/4	3/4	1/8	-1/8	15/2
3	x_1	1	0	1/2	-1/2	-1/4	1/4	5
	Z_j	3	2	0	0	-1/2	1/2	30
	$C_j - Z_j$	0	0	0	M	1/2	M-1/2	

4.

x_1 = number of F-100's

x_2 = number of F-4's

x_3 = number of A-37's

$Z_{min} = 6447x_1 + 8185x_2 + 4860x_3$

subject to:

$$1.2x_1 + 1.0x_2 + 2.0x_3 \geq 300$$
$$x_1 \leq 180$$
$$x_2 \leq 160$$
$$x_3 \leq 36$$
$$x_i \geq 0 \quad (i = 1, 2, 3)$$

C_j		6447	8185	4860	0	0	0	0	M	
	Sol. mix	x_1	x_2	x_3	s_1	s_2	s_3	s_4	A_1	b
M	A_1	6/5	1	2	-1	0	0	0	1	300
0	s_2	1	0	0	0	1	0	0	0	180
0	s_3	0	1	0	0	0	1	0	0	160
0	s_4	0	0	1	0	0	0	1	0	36
	Z_j	6/5M	M	2M	-M	0	0	0	M	300M
	$C_j - Z_j$	a	b	c	M	0	0	0	0	

a. 6447 - 6/5M

b. 8185 - M

c. 4860 - 2M

C_j		6447	8185	4860	0	0	0	0	M	
	Sol. mix	x_1	x_2	x_3	s_1	s_2	s_3	s_4	A_1	b
M	A_1	6/5	1	0	-1	0	0	-2	1	228
0	s_2	1	0	0	0	1	0	0	0	180
0	s_3	0	1	0	0	0	1	0	0	160
4860	x_3	0	0	1	0	0	0	1	0	36
	Z_j	6/5M	M	4860	-M	0	0	a	M	b
	$C_j - Z_j$	c	d	0	M	0	0	e	0	

a. -2M + 4860

b. 228M + 174,960

c. 6447 - 6/5M

d. 8185 - M

e. 2M - 4860

C_j		6447	8185	4860	0	0	0	0	M	
	Sol. mix	x_1	x_2	x_3	s_1	s_2	s_3	s_4	A_1	b
M	A_1	0	1	0	-1	-6/5	0	-2	1	12
6447	x_1	1	0	0	0	1	0	0	0	180
0	s_2	0	1	0	0	0	1	0	0	160
4860	x_3	0	0	1	0	0	0	1	0	36
	Z_j	6447	M	4860	-M	-6/5M	0	a	M	b
	$C_j - Z_j$	0	8185-M	0	M	6/5M	0	c	0	

a. -2M + 4860

b. 12M + 1,335,420

c. 2M - 4860

C_j		6447	8185	4860	0	0	0	0	M	
	Sol. mix	x_1	x_2	x_3	s_1	s_2	s_3	s_4	A_1	b
8185	x_2	0	1	0	-1	-6/5	0	-2	1	12
6447	x_1	1	0	0	0	1	0	0	0	180
0	s_3	0	0	0	1	6/5	1	2	-1	148
4860	x_3	0	0	1	0	0	0	1	0	36
	Z_j	6447	8185	4860	-8185	-3375	0	-11510	8185	a
	$C_j - Z_j$	0	0	0	8185	3375	0	11510	M-8185	

a. 1,433,640

optimal: 180 F-100's
12 F-4's
36 A-37's

minimum cost: $1,433,640 per hour.

5. (Recommend using a computer!)

x_1 = number of pounds of rolled corn

x_2 = number of pounds of barley

x_3 = number of pounds of milo

x_4 = number of pounds of cottonseed crumbles

x_5 = number of pounds of cottonseed hulls

x_6 = number of pounds of wheat mixed feed

x_7 = number of pounds of cane molasses

$Z = 5.5x + 5.3x + 5.0x + 9.0x + 3.0x + 6.0x + 2.5x$

Protein: $.089x_1 + .115x_2 + .110x_3 + .410x_4 + .040x_5 + .108x_6 + .029x_7 \geq .115$

Fat: $.035x_1 + .019x_2 + .028x_3 + .021x_4 + .044x_5 + .017x_6 \geq .025$

Fiber: $.029x_1 + .050x_2 + .020x_3 + .113x_4 + .430x_5 + .028x_6 \leq .110$

Ash: $.015x_1 + .025x_2 + .017x_3 + .064x_4 + .025x_5 + .020x_6 + .081\ _7 \leq .080$

Rolled Corn $x_1 \geq .250$

Cottonseed Hulls: $x_5 \leq .070$

Mix: $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 100$

$x \geq 0$

Cj		5.5	5.3	5	9	3	6	2.5	0	0	0	0	0	0	M	M	M	M	
	Sol. mix	x_1	x_2	x_3	x_4	x_5	x_6	x_7	s_1	s_2	s_3	s_4	s_5	s_6	A_1	A_2	A_3	A_4	b
M	A_1	.089	.115	.11	.41	.04	.108	.029	-1	0	0	0	0	0	1	0	0	0	11.5
M	A_2	.035	.019	.028	.021	.044	.017	0	0	-1	0	0	0	0	0	1	0	0	2.5
0	s_3	.029	.05	.02	.113	.43	.028	0	0	0	1	0	0	0	0	0	0	0	11
0	s_4	.015	.025	.017	.064	.025	.02	.081	0	0	0	1	0	0	0	0	0	0	8
M	A_3	1	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	0	25
0	s_6	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	7
M	A_4	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	1	100
	Zj	a	b	c	d	e	f	g	-M	-M	0	0	-M	0	M	M	M	M	139M
	Cj - Zj	h	i	j	k	l	m	n	M	M	0	0	M	0	0	0	0	0	

a. 2.124M

b. 1.134M

c. 1.138M

d. 1.431M

e. 1.084M

f. 1.125M

g. 1.029M

h. 6.5 - 2.124M

i. 5.3 - 1.134M

j. 5.1 - 1.138M

k. 9 - 1.431M

l. 3 - 1.084M

m. 6 - 1.125M

n. 2.5 - 1.029M

Cj		5.5	5.3	5	9	3	6	2.5	0	0	0	0	0	0	M	M	M	M	
	Sol. mix	x_1	x_2	x_3	x_4	x_5	x_6	x_7	s_1	s_2	s_3	s_4	s_5	s_6	A_1	A_2	A_3	A_4	b
M	A_1	0	.115	.11	.41	.04	.108	.029	-1	0	0	0	.089	0	1	0	-.089	0	9.275
M	A_2	0	.019	.028	.021	.044	.017	0	0	-1	0	0	.035	0	0	1	-.035	0	1.625
0	s_3	0	.05	.02	.113	.43	.028	0	0	0	1	0	.029	0	0	0	-.029	0	10.275
0	s_4	0	.025	.017	.064	.025	.02	.081	0	0	0	1	.015	0	0	0	-.015	0	7.625
5.5	x_1	1	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	0	25
0	s_6	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	7
M	A_4	0	1	1	1	1	1	1	0	0	0	0	1	0	0	0	-1	1	75
	Zj	5.5	a	b	c	d	e	f	-M	-M	0	0	g	0	M	M	h	M	i
	Cj - Zj	0	j	k	l	m	n	o	M	M	0	0	p	0	0	0	q	0	

a. 1.134M

b. 1.138M

c. 1.431M

d. 1.084M

e. 1.125M

f. 1.029M

g. -5.5 + 1.124M

h. 5.5 - 1.124M

i. 137.5 + 85.9M

j. 5.3 - 1.134M

k. 5 - 1.138M

l. 9 - 1.431M

m. 3 - 1.089M

n. 6 - 1.125M

o. 2.5 - 1.029M

p. 5.5 - 1.124M

q. -5.5 + 2.127M

Cj		5.5	5.3	5	9	3	6	2.5	0	0	0	0	0	0	M	M	M	M	
	Sol. mix	x_1	x_2	x_3	x_4	x_5	x_6	x_7	s_1	s_2	s_3	s_4	s_5	s_6	A_1	A_2	A_3	A_4	b
9	x_4	0	.28	.268	1	.098	.263	.071	-2.439	0	0	0	.217	0	2.439	0	-.217	0	22.622
M	A_2	0	.013	.022	0	.042	.011	-.001	.051	-1	0	0	.03	0	-.051	1	-.03	0	1.15
0	s_3	0	.018	-.01	0	.419	-.002	-.008	.276	0	1	0	.004	0	-.276	0	-.004	0	7.719
0	s_4	0	.007	0	0	.019	.003	.076	.156	0	0	1	.001	0	-.156	0	-.001	0	6.177
5.5	x_1	1	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	0	25
0	s_6	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	7
M	A_4	0	.72	.732	0	.902	.737	.929	2.439	0	0	0	.783	0	-2.439	0	-.783	1	52.378
	Zj	5.5	a	b	9	c	d	e	f	-M	0	0	g	0	h	M	i	M	j
	Zj - Cj	0	k	l	0	m	n	o	p	M	0	0	q	0	r	0	s	0	

a. 2.5 + .733M

b. 2.412 + .764M

c. .882 + .944M

d. 2.367 + .748M

e. .639 + .928M

f. -21.95 + 2.49M

g. -3.547 + .813M

h. 21.951 - 2.49M

i. 3.547 - .813M

j. 341.1 + 53.53M

k. 2.78 - .733M

l. 2.58 - .769M

m. 2.118 - .949M

n. 3.633 - .748M

o. 1.861 - .928M

p. 21.951 - 2.49M

q. 3.547 - .813M

r. -21.951 + 3.49M

s. -3.547 + .813M

Cj		5.5	5.3	5	9	3	6	2.5	0	0	0	0	0	0	M	M	M	M	
	Sol. mix	x_1	x_2	x_3	x_4	x_5	x_6	x_7	s_1	s_2	s_3	s_4	s_5	s_6	A_1	A_2	A_3	A_4	b
9	x_4	0	1	1	1	1	1	1	0	0	0	0	1	0	0	0	-1	1	75
M	A_2	0	-.002	.007	0	.023	-.004	-.02	0	-1	0	0	.014	0	0	1	-.014	-.021	.055
0	s_3	0	-.063	-.093	0	.317	-.085	.113	0	0	1	0	-.085	0	0	0	.085	-.113	1.792
0	s_4	0	-.039	-.047	0	-.039	-.044	.017	0	0	0	1	-.049	0	0	0	.049	-.064	2.827
5.5	x_1	1	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	0	25
0	s_6	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	7
0	s_1	0	.295	.300	0	.37	.302	.381	1	0	0	0	.321	0	-1	0	-.321	.41	21.475
	Zj	5.5	a	b	9	c	d	e	0	-M	0	0	f	0	0	M	g	h	i
	Cj - Zj	0	j	k	0	l	m	n	0	M	0	0	p	0	M	0	q	r	

a. 9 - .002M

b. 9 + .007M

c. 9 + .023M

d. 9 - .004M

e. 9 - .02M

f. 3.5 + .014M

g. -3.5 - .014M

h. 9 - .021M

i. 812.5 + .055M

j. -3.7 + .002M

k. -4 - .007M

l. -6 - .023M

m. -3 + .004M

n. -6.5 + .02M

p. -3.5 - .014M

q. 3.5 + 1.014M

r. -9 + 1.021M

Cj		5.5	5.3	5	9	3	6	2.5	0	0	0	0	0	0	M	M	M	M	
	Sol. mix	x_1	x_2	x_3	x_4	x_5	x_6	x_7	s_1	s_2	s_3	s_4	s_5	s_6	A_1	A_2	A_3	A_4	b
9	x_4	0	1.087	.696	1	0	1.174	1.87	0	43.478	0	0	.391	0	0	-43.478	-.391	1.913	72.609
3	x_5	0	-.087	.324	0	1	-.174	-.87	0	-43.478	0	0	.609	0	0	43.478	-.609	-.913	2.391
0	s_3	0	-.035	-.189	0	0	-.03	.163	0	13.783	1	0	-.278	0	0	-13.783	.278	.176	1.034
0	s_4	0	-.042	-.035	0	0	-.051	-.017	0	-1.696	0	1	-.025	0	0	1.696	.025	-1	2.92
5.5	x_1	1	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	0	25
0	s_6	0	.087	-.304	0	0	.174	.87	0	43.478	0	0	-.609	1	0	-43.478	.609	.913	4.609
0	s_1	0	.327	.188	0	0	.366	.703	1	16.087	0	0	.096	0	-1	-16.087	-.096	.748	20.59
	Zj	5.5	9.522	7.176	9	3	10.044	14.22	0	260.868	0	0	-.154	0	0	-260.868	.154	14.478	798.15
	Cj - Zj	0	-4.222	-2.176	0	0	-4.044	-11.72	0	-260.868	0	0	.154	0	M	a	b	c	

a. M + 260.868

b. M - .154

c. M - 14.478

Cj		5.5	5.3	5	9	3	6	2.5	0	0	0	0	0	0	M	M	M	M	
	Sol. mix	x_1	x_2	x_3	x_4	x_5	x_6	x_7	s_1	s_2	s_3	s_4	s_5	s_6	A_1	A_2	A_3	A_4	b
9	x_4	0	1.217	1.305	1	0	1.261	1.348	0	0	-3.174	0	1.261	0	0	0	-1.261	1.348	69.348
3	x_5	0	-2.17	-.305	0	1	-.261	-.348	0	0	3.174	0	-.261	0	0	0	.261	-.348	5.652
0	s_2	0	-.003	-.014	0	0	-.002	.012	0	1	.073	0	-.02	0	0	-1	.02	.013	.075
0	s_4	0	-.047	-.059	0	0	-.054	.003	0	0	.124	1	-.059	0	0	0	.059	-.078	3.047
5.5	x_1	1	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	0	25
0	s_6	0	.217	.305	0	0	.261	.348	0	0	-3.174	0	.261	1	0	0	-.261	.348	1.348
0	s_1	0	.375	.413	0	0	.398	.510	1	0	-1.174	0	.481	0	-1	0	-.418	.539	19.383
	Zj	5.5	10.302	10.83	9	3	10.566	11.088	0	0	-19.044	0	5.066	0	0	0	-5.066	11.088	778.59
	Cj - zj	0	-5.002	-5.83	0	0	-4.566	-8.588	0	0	19.044	0	-5.066	0	M	M	a	b	

a. M + 5.066

b. M - 11.088

Cj		5.5	5.3	5	9	3	6	2.5	0	0	0	0	0	0	M	M	M	M	
	Sol. mix	x_1	x_2	x_3	x_4	x_5	x_6	x_7	s_1	s_2	s_3	s_4	s_5	s_6	A_1	A_2	A_3	A_4	b
9	x_4	0	.376	.124	1	0	.25	0	0	0	9.121	0	.25	-3.874	0	0	-.25	0	64.126
3	x_5	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	7
0	s_2	0	-.01	-.025	0	0	-.011	0	0	1	.182	0	-.029	-.034	0	-1	.029	.001	.029
0	s_4	0	-.049	-.062	0	0	-.056	0	0	0	.151	1	-.061	-.009	0	0	.061	-.081	3.035
5.5	x_1	1	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	0	25
2.5	x_7	0	.624	.876	0	0	.75	1	0	0	-9.121	0	.75	2.874	0	0	-.75	1	3.874
0	s_1	0	.057	-.034	0	0	.016	0	1	0	3.478	0	.036	-1.466	-1	0	-.036	.029	17.407
	Zj	5.5	4.944	3.306	9	3	4.125	2.5	0	0	59.287	0	-1.375	-24.681	0	0	1.375	2.5	745.32
	Cj - Zj	0	.356	1.694	0	0	1.875	0	0	0	-59.287	0	1.375	24.681	M	M	M-1.375	M-2.5	

Cj	Sol. mix	x_1	x_2	x_3	x_4	x_5	x_6	x_7	s_1	s_2	s_3	s_4	s_5	s_6	A_1	A_2	A_3	A_4	b
		5.5	5.3	5	9	3	6	2.5	0	0	0	0	0	0	M	M	M	M	
9	x_4	0	.878	1.374	1	0	.797	0	0	-50.12	0	0	1.7	-2.168	0	50.12	-1.7	-.046	62.676
3	x_5	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	7
0	s_3	0	-.055	-.137	0	0	-.06	0	0	5.495	1	0	-.159	-.187	0	-5.495	.159	.005	.159
0	s_4	0	-.041	-.041	0	0	-.047	0	0	-.83	0	1	-.037	.019	0	.83	.037	-.082	3.011
5.5	x_1	1	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	0	25
2.5	x_7	0	.122	-.374	0	0	.203	1	0	50.12	0	0	-.7	1.168	0	-50.12	.7	1.046	5.324
0	s_1	0	.248	.442	0	0	.225	0	1	-19.112	0	0	.589	-.816	-1	19.112	-.589	.012	16.854
	Z_j	5.5	8.207	11.431	9	3	7.681	2.5	0	-325.78	0	0	8.05	-13.592	0	325.78	-8.05	3.029	735.89
	$C_j - Z_j$	0	-2.907	-6.431	0	0	-1.681	0	0	325.78	0	0	-8.05	13.590	M	a	b	c	

a. M - 325.78

b. M + 8.05

c. M - 3.029

Cj		5.5	5.3	5	9	3	6	2.5	0	0	0	0	0	0	M	M	M	M	
	Sol. mix	x_1	x_2	x_3	x_4	x_5	x_6	x_7	s_1	s_2	s_3	s_4	s_5	s_6	A_1	A_2	A_3	A_4	b
9	x_4	0	.162	.099	1	0	.148	0	-2.887	5.042	0	0	0	.187	2.887	-5.042	0	-.08	14.122
3	x_5	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	7
0	s_3	0	.012	-.018	0	0	.001	0	.27	.336	1	0	0	-.407	-.27	-.336	0	.008	4.709
0	s_4	0	-.025	-.013	0	0	-.033	0	.063	-2.031	0	1	0	-.032	-.063	2.031	0	-.081	4.07
5.5	x_1	1	.421	.75	0	0	.382	0	1.698	-32.448	0	0	0	-1.395	-1.698	32.448	0	.02	53.615
2.5	x_7	0	.417	.151	0	0	.47	1	1.189	27.406	0	0	0	.199	-1.189	-27.406	0	1.06	25.355
0	s_5	0	.421	.75	0	0	.382	0	1.698	-32.448	0	0	1	-1.385	-1.698	32.498	-1	.02	28.615
	Zj	5.5	4.816	5.394	9	3	4.608	2.5	-13.672	a	0	0	0	-2.437	13.672	64.571	0	2.04	506.37
	Cj - Zj	0	.484	-.394	0	0	1.392	0	13.672	b	0	0	0	2.437	c	d	M	M-2.04	

a. -64.571

b. 64.571

c. M - 13.672

d. M - 64.571

Cj		5.5	5.3	5	9	3	6	2.5	0	0	0	0	0	0	M	M	M	M	
	Sol. mix	x_1	x_2	x_3	x_4	x_5	x_6	x_7	s_1	s_2	s_3	s_4	s_5	s_6	A_1	A_2	A_3	A_4	b
9	x_4	0	.106	0	1	0	.098	0	-3.111	9.325	0	0	-.132	.373	3.111	-9.325	.132	-.082	10.345
3	x_5	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	7
0	s_3	0	.022	0	0	0	.01	0	.311	-.443	1	0	.024	-.44	-.311	-.443	-.024	.008	5.396
0	s_4	0	-.018	0	0	0	-.021	0	.092	-2.593	0	1	.017	-.056	-.092	2.593	-.017	-.081	4.566
5.5	x_1	1	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	0	25
2.5	x_7	0	.332	0	0	0	.393	1	.847	33.939	0	0	-.201	.478	-.847	a	.201	1.056	19.594
5	x_3	0	.561	1	0	0	.509	0	2.264	b	0	0	1.333	-1.847	c	d	-1.333	.027	38.153
	Zj	5.5	4.443	5	9	3	4.41	2.5	e	f	0	0	-.526	-1.683	g	h	.526	7.037	491.36
	Cj - Zj	0	.857	0	0	0	1.59	0	i	j	0	0	.526	1.683	k	l	m	n	

a. -33.939

b. -43.264

c. -2.264

d. 43.264

e. -14.562

f. -47.548

g. 14.562

h. 47.548

i. 14.562

j. 47.548

k. M - 14.562

l. M - 47.548

m. M - .526

n. M - 2.037

Due to rounding error, the following is the final tableau, rounded to three places, which was calculated on a computer.

Cj		5.5	5.3	5	9	3	6	2.5	0	0	0	0	0	0	M	M	M	M	
	Sol. mix	x_1	x_2	x_3	x_4	x_5	x_6	x_7	s_1	s_2	s_3	s_4	s_5	s_6	A_1	A_2	A_3	A_4	b
9	x_4	0	.097	0	1	0	.093	0	-3.123	9.033	0	0	-.129	.363	3.123	-9.033	.129	-.091	10.033
3	x_5	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	7
0	s_3	0	.027	0	0	0	.007	0	.306	-.171	1	0	.017	-.434	-.306	.171	-.017	.009	5.341
0	s_4	0	-.016	0	0	0	-.025	0	.097	-2.566	0	1	.018	-.056	-.097	2.566	-.018	-.078	4.641
5.5	x_1	1	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1	0	25
2.5	x_7	0	.297	0	0	0	.370	1	.781	33.956	0	0	-.218	.481	-.781	-33.456	.218	1.023	18.456
5	x_3	0	.606	1	0	0	.537	0	2.342	-42.489	0	0	1.347	-1.844	-2.342	42.489	-1.347	.068	39.511
	Zj	5.5	4.645	5	9	3	4.449	2.5	a	b	0	0	-.471	-1.749	c	d	e	f	g
	Cj - Zj	0	.655	0	0	0	1.551	0	h	i	0	0	.471	1.749	j	k	l	m	

a. -14.442

b. -47.508

c. 14.442

d. 47.508

e. .471

f. 2.081

g. 492.493

h. 14.442

i. 47.508

j. M - 14.442

k. M - 47.508

l. M - .471

m. M - 2.081

6. x_1 = acid loving soil

x_2 = potting soil

$Z_{min} = 4.69x_1 + 3.11x_2$

subject to:

$$\begin{aligned} x_1 + x_2 &\geq 20 \\ x_1 &\geq 10 \\ x_2 &\geq 5 \\ 0.23x_1 + 0.40x_2 &\leq 6 \\ 0.10x_1 + 0.42x_2 &\leq 10 \\ x_i &\geq 0 \end{aligned}$$

Cj		4.69	3.11	0	0	0	0	0	M	M	M	
	Sol. mix	x_1	x_2	s_1	s_2	s_3	s_4	s_5	A_1	A_2	A_3	b
M	A_1	1	1	-1	0	0	0	0	1	0	0	20
M	A_2	1	0	0	-1	0	0	0	0	1	0	10
M	A_3	0	1	0	0	-1	0	0	0	0	1	5
0	s_4	.23	.40	0	0	0	1	0	0	0	0	6
0	s_5	.10	.42	0	0	0	0	1	0	0	0	10
	Zj	2M	2M	-M	-M	-M	0	0	M	M	M	35M
	Cj - Zj	a	b	M	M	M	0	0	0	0	0	

a. 4.69 - 2M

b. 3.11 - 2M

Cj		4.69	3.11	0	0	0	0	0	M	M	M	
	Sol. mix	x_1	x_2	s_1	s_2	s_3	s_4	s_5	A_1	A_2	A_3	b
M	A_1	1	0	-1	0	1	0	0	1	0	-1	15
M	A_2	1	0	0	-1	0	0	0	0	1	0	10
3.11	x_2	0	1	0	0	-1	0	0	0	0	1	5
0	s_4	.23	0	0	0	.4	1	0	0	0	-.4	5
0	s_5	.1	0	0	0	.42	0	1	0	0	-.42	7.9
	Zj	2M	3.11	-M	-M	-3.11+M	0	0	M	M	3.11-M	15.55+25M
	Cj - Zj	4.69-2M	0	M	M	3.11-M	0	0	0	0	-3.11+2M	

Cj		4.69	3.11	0	0	0	0	0	M	M	M	
	Sol. mix	x_1	x_2	s_1	s_2	s_3	s_4	s_5	A_1	A_2	A_3	b
M	A_1	0	0	-1	1	1	0	0	1	-1	-1	5
4.69	x_1	1	0	0	-1	0	0	0	0	1	0	10
3.11	x_2	0	1	0	0	-1	0	0	0	0	1	5
0	s_4	0	0	0	.23	.4	1	0	0	-.23	-.4	1.7
0	s_5	0	0	0	.1	.42	0	1	0	-.1	-.42	6.9
	Zj	4.69	3.11	-M	-4.69+M	-3.11+M	0	0	M	4.69-M	3.11-M	62.45+5M
		0	0	M	4.69-M	3.11-M	0	0	0	-4.69-2M	-3.11+2M	

Cj		4.69	3.11	0	0	0	0	0	M	M	M	
	Sol. mix	x_1	x_2	s_1	s_2	s_3	s_4	s_5	A_1	A_2	A_3	b
M	A_1	0	0	-1	.425	0	-2.5	0	1	-.425	0	.75
4.69	x_1	1	0	0	-1	0	0	0	0	1	0	10
3.11	x_2	0	1	0	.575	0	2.5	0	0	-.575	0	9.25
0	s_3	0	0	0	.575	1	2.5	0	0	-.575	-1	4.25
0	s_5	0	0	0	-.142	0	-1.05	1	0	.142	0	5.12
	Zj	4.69	3.11	-M	a	0	b	0	M	c	0	75.67+.75M
	Cj - Zj	0	0	M	d	0	e	0	0	f	M	

a. -2.902+.425M b. 7.775-2.5M c. 2.902-.425M
d. 2.902-.425M e. -7.775+2.5M f. -2.902+1.425M

Cj		4.69	3.11	0	0	0	0	0	M	M	M	
	Sol. mix	x_1	x_2	s_1	s_2	s_3	s_4	s_5	A_1	A_2	A_3	b
0	s_2	0	0	-2.353	1	0	-5.882	0	2.353	-1	0	1.765
4.69	x_1	1	0	-2.353	0	0	-5.882	0	2.353	0	0	11.765
3.11	x_2	0	1	1.353	0	0	5.882	0	-1.353	0	0	8.235
0	s_3	0	0	1.353	0	1	5.882	0	-1.353	0	-1	3.235
0	s_5	0	0	-.334	0	0	-1.885	1	.334	0	0	5.371
	Zj	4.69	3.11	a	0	0	-9.294	0	6.828	0	0	80.79
	Cj - Zj	0	0	b	0	0	9.294	0	M-6.828	M	M	

a. -6.828

b. 6.828

Exercise 5.6, pp. 272-273.

1. $Z_{min} = 10y_1 - 2y_2 + 20y_3$

subject to:

$$y_1 + y_3 \geq 2$$

$$y_1 - y_2 + 2y_3 \geq 1$$

$$y_i \geq 0 \quad (i = 1, 2, 3)$$

2. $Z_{min} = 20y_1 - 10y_2 + 30y_3 - 30y_4$

subject to:

$$y_1 - y_2 + 3y_3 - 3y_4 \geq 1$$

$$2y_1 - 2y_2 + 2y_3 - 2y_4 \geq 5$$

$$y_i \geq 0 \quad (1, 2, 3, 4)$$

3. $Z_{min} = 20y_1 + 10y_2 - 2y_3 + 3y_4 - 10y_5 + 20y_6$

subject to:

$$4y_1 + y_2 - y_3 + 2y_6 \geq 2$$

$$-5y_1 + 2y_2 + y_4 - y_5 + y_6 \geq 1$$

$$y_i \geq 0 \quad \_i = 1, 2, 3, 4, 5, 6)$$

4. $Z_{max} = 20y_1 + 10y_2 + 15y_3$

subject to:

$$\begin{aligned} 10y_1 + y_2 + y_3 &\le 2 \\ y_1 + y_2 + 3y_3 &\le 2 \\ y_i &\ge 0 \quad (i = 1, 2, 3) \end{aligned}$$

5. $Z_{max} = 20y_1 - 10y_2 + 5y_3 + 6y_4 - 6y_5$

subject to:

$$\begin{aligned} y_1 - 2y_2 + y_3 + y_4 - y_5 &\le 3 \\ 2y_1 + 4y_3 + 2y_4 - 2y_5 &\le 2 \\ -3y_1 - y_2 + 2y_3 &\le 1 \\ y_i &\ge 0 \quad (i = 1, 2, 3, 4, 5) \end{aligned}$$

CHAPTER 6

NON-LINEAR POLYNOMIAL FUNCTIONS

The quadratic function is the mainstay of this chapter. Polynomial functions are discussed with the concepts of critical point, concavity, and x-intercept being stressed.

By developing skills using the quadratic equation, students will be ready to comprehend the applications discussed in differential calculus. The coefficient of the x^2 term is brought to the students' attention on page 282.

Imaginary roots are often perplexing--the graph in example 9 on page 287 shows how this occurs.

Exercise 6.3, pp. 287-289.

Formula 6.5:

$$x = -\left(\frac{b}{2a}\right) \qquad y = -\left(\frac{b^2 - 4ac}{4a}\right)$$

1. $x = -\left(\frac{(2)}{2(1)}\right) = -1 \qquad y = -\left(\frac{(2)^2 - 4(1)(5)}{4(1)}\right) = 4$ (-1,4)

2. $x = -\left(\frac{(2)}{2(-1)}\right) = 1 \qquad y = -\left(\frac{(2)^2 - 4(-1)(5)}{4(-1)}\right) = 6$ (1,6)

3. $x = -\left(\frac{(0)}{2(3)}\right) = 0 \qquad y = -\left(\frac{(0)^2 - 4(3)(2)}{4(3)}\right) = -2$ (0,-2)

4. $x = -\left(\frac{(7)}{2(1)}\right) = 3.5 \qquad y = -\left(\frac{(7)^2 - 4(1)(0)}{4(1)}\right) = -12.25$ (-3.5,-12,25)

5. $x = -\left(\frac{(1/2)}{2(1)}\right) = 0.25 \qquad y = -\left(\frac{(1/2)^2 - 4(-1)(3)}{4(-1)}\right) = 3.0625$ (0.25, 3.0625)

6. $x = -\left(\frac{(0)}{2(5)}\right) = 0 \qquad y = -\left(\frac{(0)^2 - 4(5)(0)}{4(5)}\right) = 0$ (0,0)

7. $x = -\left(\frac{(2/3)}{2(-3)}\right) = 0.1111 \qquad y = \left(\frac{(2/3)^2 - 4(-3)(-1/3)}{4(-3)}\right) = -0.2963$

8. f(x) is a straight line

Formula (6.6): $a(x - h)^2 = (y - k)$

9. $y = x^2 - 8x + 12$

$x^2 - 8x + (8/2)^2 = y - 12 + (8/2)^2$

$(x - 4)^2 = y + 4$

$1(x - 4)^2 = [y - (-4)$ (4,-4)

10. $y = 5x^2 + 4x - 12$

$x^2 + 4/5x + (4/10)^2 = y + 12 + (4/10)^2$

$(x + 4/10)^2 = y + 12.16$

$1[x - (-0.4)] = [y - (-12.16)]$ (-0.4, -12.16)

11. $y = x^2$

$(1)(x - 0)^2 = y - 0$ (0,0)

12. $y = -x^2 - 120x$

$(-1)[x^2 + 120x + (120/2)^2] = y + (120/2)^2$

$(-1)(x + 60)^2 = y + 3600$

$(-1)[x - (-60)]^2 = [y - (-3600)]$ (-60,-3600)

13. $y = x^2 + 2x + 5$

$x^2 + 2x + (2/2)^2 = y - 5 + (2/2)^2$

$(x + 1)^2 = y - 4$

$[x - (-1)]^2 = (y - (4)]$ (-1,4)

14. $y = x^2 - 10$

$(x + 0)^2 = y + 10$

$(x + 0)^2 = [y - (-10)]$ (0,-10)

15. f(x) is a straight line

16. $x^2 - 8x + 12 = 0$

$(x - 2)(x - 6) = 0$

$x = 2,6$

Note: The problems in this segment are a review and may be omitted.

17. $x^2 + 2x + 5 = 0$

$x^2 + 2x + (2/2)^2 = -5 + (2/2)^2$

$(x = 1)^2 = -4$

$x = 1 = (-4)^{1/2}$

$x = 1 + 2i,\ 1 - 2i$ (solution is imaginary)

18. $x^2 - 4x + 4 = 0$

$(x - 2)^2 = 0$

$x = 2,2$

19. $-3x^2 - 2x + 8 = 0$

$x^2 + 2/3x = 8/3$

$x + 2/3x + (2/6)^2 = 8/3 + (2/6)^2$

$(x + 1/3)^2 = 25/9$

$x = -2,\ +1.33$

20. $x^2 - 5 = 0$

$x^2 = 5$

$x = \pm 2.236$

21. $-x^2 - 5 = 0$

$x^2 = -5$

$x = \pm 2.236i$

Quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

22. $x = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} = -1 \pm 2i$

23. $x = \dfrac{0 \pm \sqrt{0^2 - 4(25)(-100)}}{2(25)} = \pm 2$

24. $x = \dfrac{+8 \pm \sqrt{(-8)^2 - 4(1)(15)}}{2(1)} = 3,5$

25. $x = 0,0$

26. $x = \frac{+8 \pm \sqrt{(-8)^2 - 4(1)(12)}}{2(1)} = 2,6$

27. x = -8, -2

28. x = 3, -2

29. x = 5, 2

30. x = 1.23, -1.897

31. x = 0, .67

32. x = (-2 ± 11.66i)/-14

33. x - 1, -1

34. x = 0, -1.5

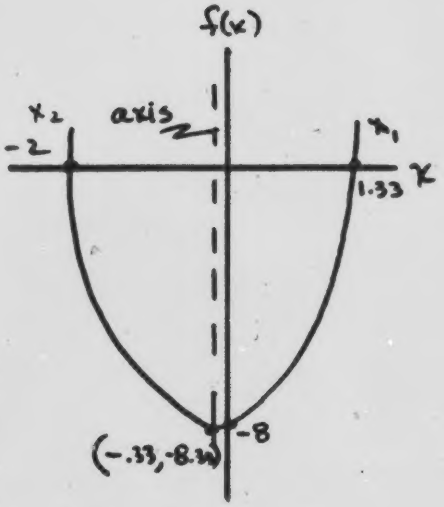
35. x = -3, 3

36.

("a" is positive-upward)

vertex (-.33,-8.33)

y-intercept (-8)



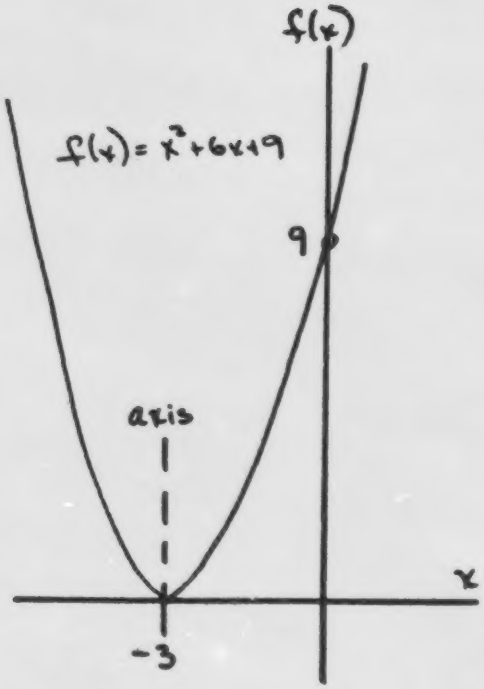
$f(x) = 3x^2 - 2x - 8$

37.

"a" is positive-upward

y-intercept (+9)

vertex (-3,0)

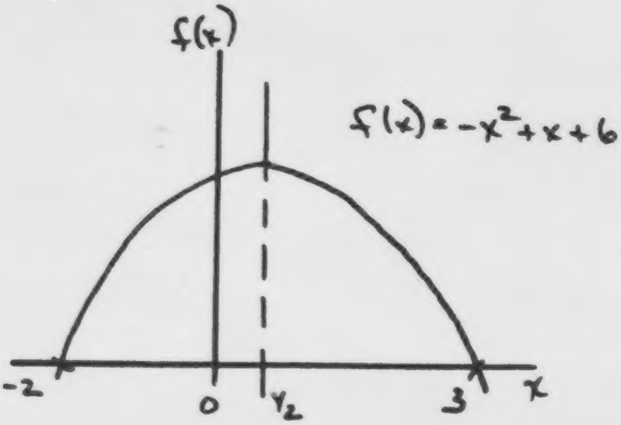


38.

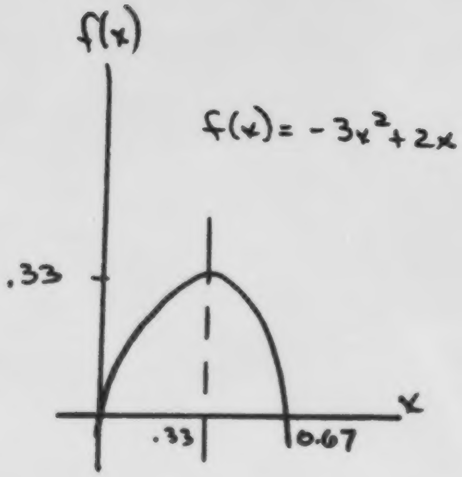
"a" is negative-downward

y-intercept (6)

vertex (.5,6.2)



39.

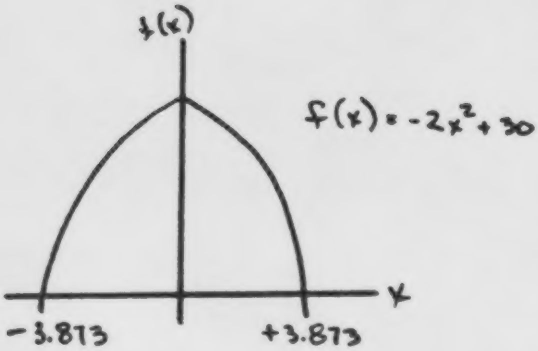


"a" is negative-downward

y-intercept (0)

vertex (.33,.33)

40.

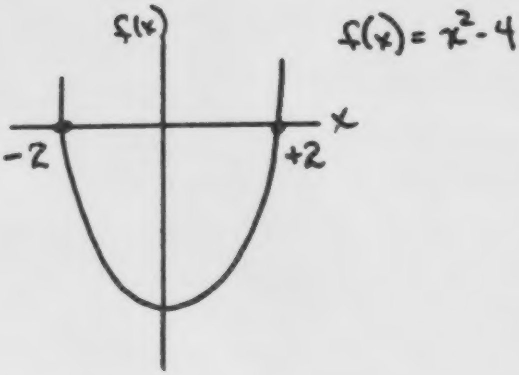


"a" is negative-downward

y-intercept (35)

vertex (0,30)

41.



"a" positive-upwards

y-intercept (-16)

vertex (0,-16)

42. $P(x) = -.2x^2 + 5x - 20$

$x = -b/2a$

$x = (-5)/2(-.2)$

$x = 12.5$

$$x = \frac{-5 \pm \sqrt{5^2 - 4(-.2)(-20)}}{4(-.2)}$$

$$= \frac{-5 \pm 3}{-.4}$$

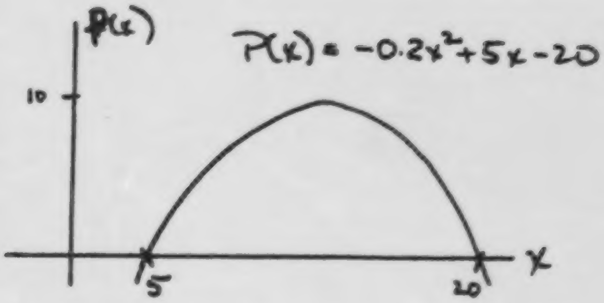
$x_1 = 5$

$x_2 = 20$

a. 12.5 cards

b. breakeven
$x_1 = 5$

$x_2 = 20$



43. $x_1 = 0$

$x_2 = 50$

$h = (50 - 0)/2$

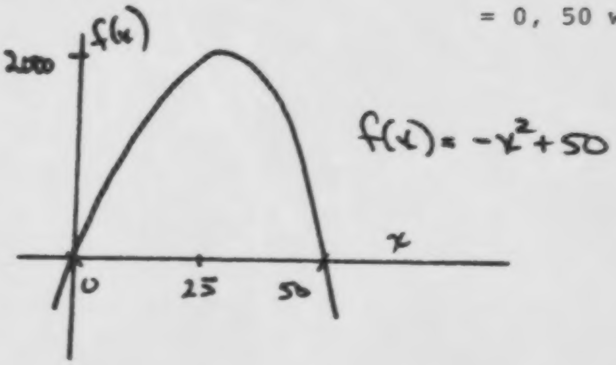
$= 25$

$h = -b/2a = 25$

$h = -50/(2)(-1) = 25$

$$x = \frac{-50 \pm \sqrt{50^2 - 4(-1)c}}{2(-1)}$$

$= 0, 50$ when $c = 0$



44. vertex (200, 600)

$x_1 = 0$

$x_2 = (x - 0)/2 = 200$

$x_2 = 400$

$a(0)^2 + b(0) + c = 0$ $\qquad c = 0$

$a(400)^2 + b(400) + c = 0$

$a(200)^2 + b(200) + c = 600$

$160{,}000\,A + 400B + c = 0$

$40{,}000\,A + 200B + C = 600$

$160{,}000 + 400B + 0 = 0$

$\underline{80{,}000 + 400B + 0 = 1200}$

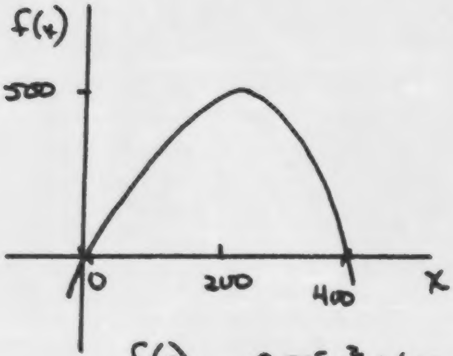
$80{,}000A = -1200$

$A = -.015$

$(-.015)(160{,}000) + 400B + 0 = 0$

$400B = 2400$

$B = 6$



$f(x) = -0.015x^2 + 6x$

45. $O(x) = 6x^2 - 24x + 30$

$h = (-24)/2(6) = 2$

$$k = -\frac{(-24)^2 - 4(6)(30)}{4(6)} = 6$$

vertex (2,6)

a. increase after 2

b. $O(0) = 6(0)^2 - 24(0) + 30$

$= 30$

46. $P(x) = -x^2 + 90x - 1400 - (x^2 - 90x + 2400)$

$P(x) = -2x^2 + 180x - 3800$

$$x = \frac{-180 \pm \sqrt{180^2 - 4(-2)(-3800)}}{2(-2)}$$

$$= \frac{-180 \pm \sqrt{200}}{-4}$$

$x_1 = 33.82$

$x_2 = 56.18$

$n = -[180/2(-2)]$

$= 45$

$$k = -\frac{180^2 - 4(-2)(-3800)}{4(-2)}$$

$= 250$

a. breakeven 33.82, 56.18

b. $x = 45$

c. $y = \$250$

CHAPTER 7

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Exponential and logarithmic functions are discussed. Exponential functions are graphed. The concept of a curve's being asymptotic to the x-axis is an important aspect of this section.

Progressions are introduced with tabular presentation and graphics on page 299. The problems in exercise 7.1 show the student the applications of both exponential functions and progressions.

Logarithms are presented primarily to have students learn the rules on pages 304-305. These rules are invaluable in differential calculus.

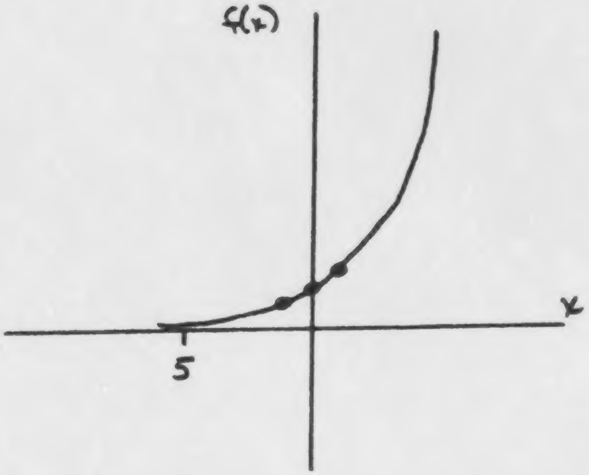
Many texts show complicated proofs to explains that exponential functions and logarithmic functions are inverse functions. Example 18 on pages 309-310 displays the inverse relationships so that it is visually obvious.

Exercise 7.1, pp. 300-301.

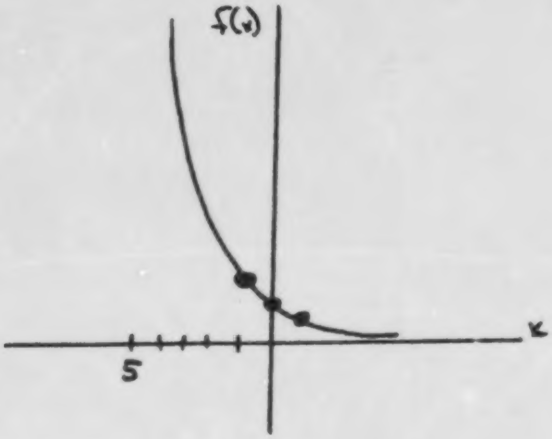
Given $f(x) = b^x$ $(b > 0,\ b \neq 1)$

The essential points are (1, 1/b), (0, 1), (1,b)

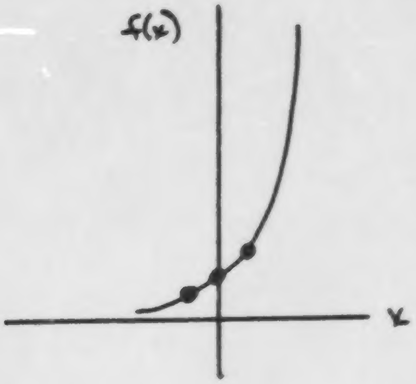
1. $f(x) = 2^x$ Thus (-1, 1/2), (0, 1), (1, 2)



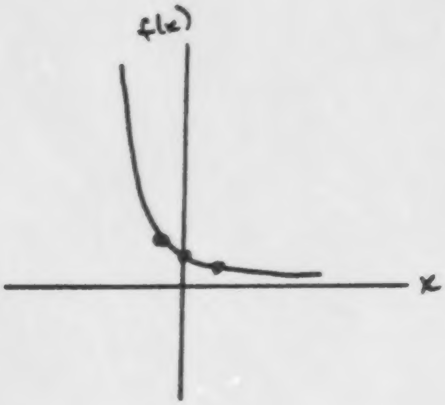
2. $f(x) = 2^{-x}$ Thus, (-1, 2), (0, 1), (1, 1/2)



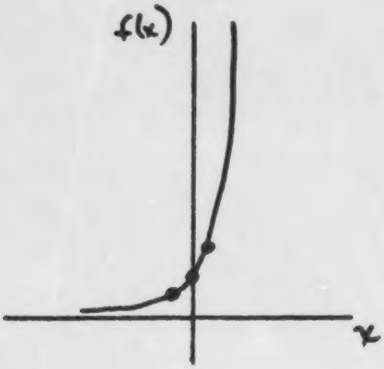
3. $f(x) = (1/3)^{-x}$, thus (-1, 1/3), (0, 1), (1, 3)



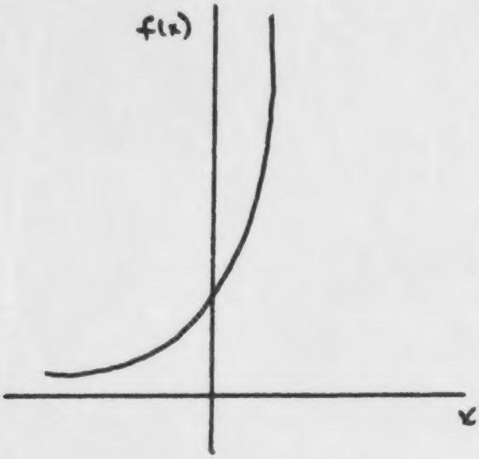
4. $f(x) = e^{-x}$, thus (-1, e), (0, 1), (1, 1/e)



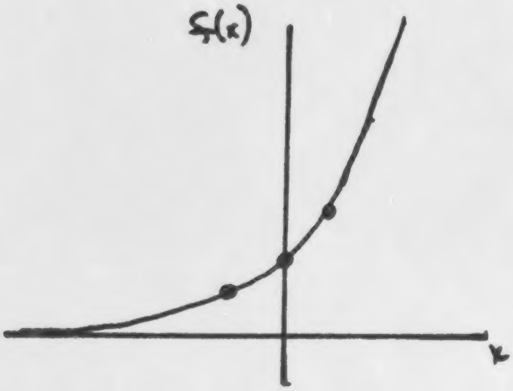
5. $f(x) = 5e^x$, thus (-1, 5/e), (0, 1), (1, 5e).



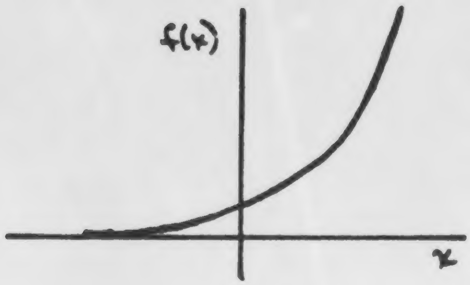
6. $f(x) = 10*2^x$, thus (-1, 5), (0, 10), (1, 20)



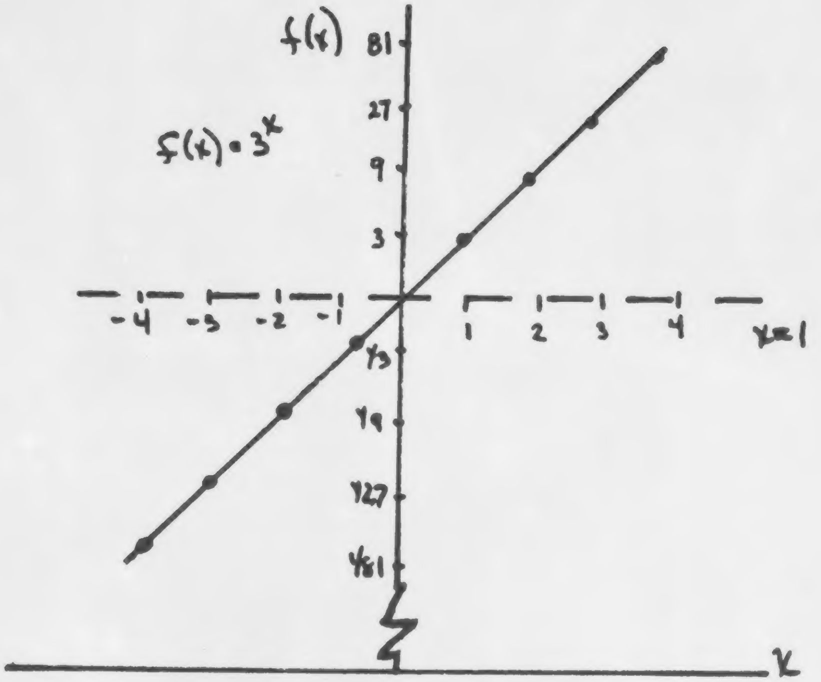
7. $f(x) = 0.4^{-x}$, thus (-1, 0.4), (0, 1), (1, 2.5)



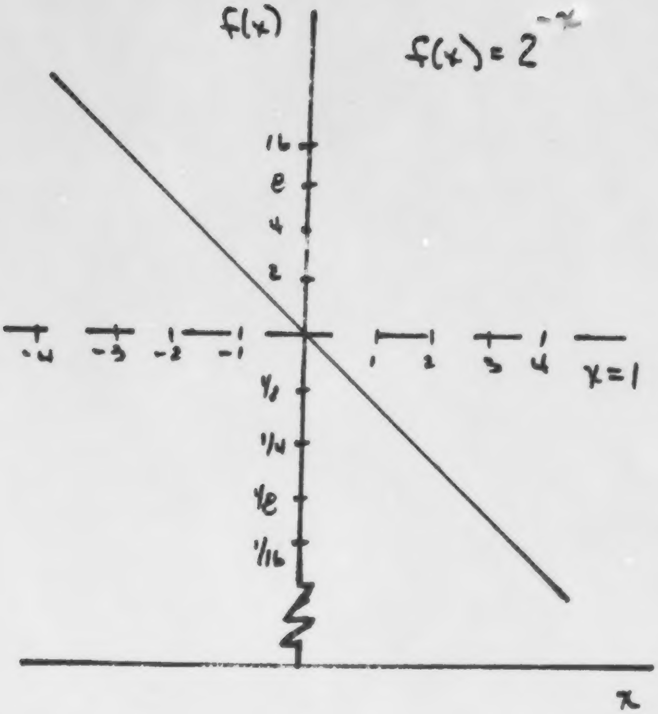
8. $f(x) = 1.5^x$, thus (-1, 0.67), (0, 1), (1, 1.5)



9. $f(x) = 3^x$



10. $f(x) = 2^{-x}$



11. $f(x) = 117(\text{rate})^x$

(a) $f(x) = 117(1.127)^5 = \$212.72$

(b) $f(x) = 117(1.156)^{-5} = \$\ 56.68$

12. $f(x) = 250(1 - .20)^x = 2.50(.80)^x$

using 1970 as base year, then for 1971, x = 1

1979, x = 9 and $f(x) = 250(.80)^9 = \$33.55$

1980, x = 10 $f(x) = 250(.80)^{10} = \$26.84$

1984, x = 14 $f(x) = 250(.80)^{14} = \$11.00$

1985, x = 15 $f(x) = 250(.80)^{15} = \$\ 8.80$

13. $f(x) = 1{,}923{,}295(1.059)^x$

1985, x = 15, f(15) - 4,544, 491.59

1990, x = 20, f(20) = 6,052,922.38

1995, x = 25, f(25) = 8,062,003.92

14. $f(x) = 700{,}000(1.08)^x$

1983, x = 13, f(13) = 1,902,736.61

1985, x = 15, f(15) = 2,220,518.38

1990, $x = 20$, $f(20) = 3{,}262{,}670.00$

15. (a) $f(x) = (0.005)2^x$

(b) $f(30) = \$5{,}368{,}709.12$

(c) Go for the pennies; you will make more.

(d) $(0.005)2^x = 1{,}000{,}000$

$$2^x = 200{,}000{,}000$$

$$x \ln 2 = \ln (200{,}000{,}000)$$

$$x = 19.11383/0.69315 = 27.6$$

The 28th day

(e)

	x$	day	total	
	0.01	1	0.01	
	0.02	2	0.03	
	0.04	3	0.07	
	0.08	4	0.15	
2x	0.16	5	0.31	2x - 1
	0.32	6	0.63	
	0.64	7	1.27	
	1.28	8	2.55	
	2.56	9	(2x - 1)	
	2x	10		
	↓	↓	↓	
	5,368,709.12	30	$10,737,418.23	

Exercise 7.2, pp. 313-315.

1. $7 = 3^x \quad x = \log_3 7 \quad \dfrac{\log_{10} 7}{\log_{10} 3} = \dfrac{0.84510}{0.47712} = 1.77125$

2. $14 = 2^x \quad x = \log_2 14 \quad \dfrac{\log_{10} 14}{\log_{10} 2} = \dfrac{1.14613}{0.30103} = 3.80736$

3. $5 = 0.3^x \quad x = \log_{0.3} 5 \quad \dfrac{\log_{10} 5}{\log_{10} 0.3} \quad \dfrac{0.69897}{0.47712 - 1} = -1.3368$

4. $0.2 = .4^x \quad x = \log_{0.4}0.2 \quad \dfrac{\log_{10}0.2}{\log_{10}0.4} = \dfrac{0.30103 - 1}{0.60206 - 1} = 1.75647$

5. $87 = 5^x \quad x = \log_5 87 \quad \dfrac{\log_{10}87}{\log_{10}5} = \dfrac{1.93952}{0.69897} = 2.77483$

6. $f(x) = 2^x \quad e^{x\log_e 2} = e^{x \ln 2}$

7. $f(x) = .4^x \quad e^{x\log_e .4} = e^{x \ln 0.4}$

8. $f(x) = 5^x \quad e^{x\log_e 5} = e^{x \ln 5}$

9. $f(x) = k^x \quad e^{x\log_e k} = e^{x \ln k}$

10. $f(x) = \log_2 9 \quad \dfrac{\log_{10}9}{\log_{10}2} = \dfrac{0.95424}{0.30103} = 3.16992$

11. $f(x) = \log_5 15 \quad \dfrac{\log_{10}15}{\log_{10}5} = \dfrac{1.17609}{0.69897} = 1.68260$

12. $f(x) = \log_3 2 \quad \dfrac{\log_{10}2}{\log_{10}3} = \dfrac{0.30103}{0.47712} = 0.63093$

13. $f(x) = \log_e 4 \quad \dfrac{\log_{10}4}{\log_{10}e} = \dfrac{0.60206}{0.43429} = 1.38631$

14. $f(x) - \log_{14} 2 \quad \dfrac{\log_{10}2}{\log_{10}14} = \dfrac{0.30103}{1.14613} = 0.26265$

15. $f(x) = \log_7 21 \quad \dfrac{\log_{10}21}{\log_{10}7} = \dfrac{1.32222}{0.84510} = 1.56457$

16. $921 = 9.21\times10^2$

17. $.02001 = 2.001\times10^{-2}$

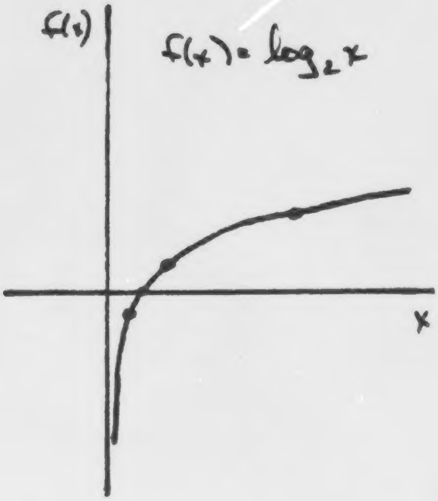
18. $82.1 = 8.21\times10^1$

19. $1005.2 = 1.00052\times10^4$

20. $.002 = 2\times10^{-3}$

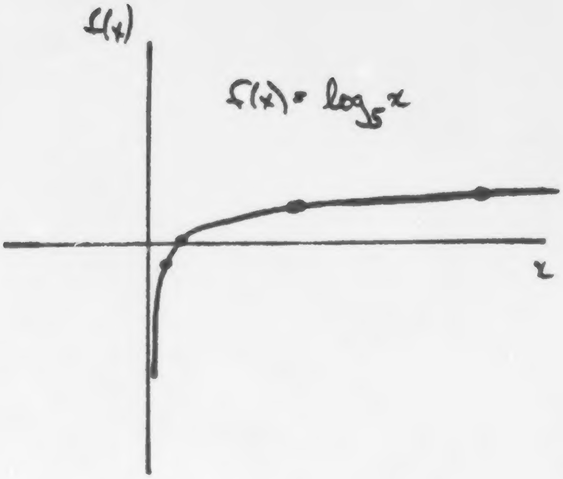
21. $f(x) = \log_2 x$ $x = 2^{f(x)}$

x	2	1	0.5	3
f(x)	1	0	-1	8



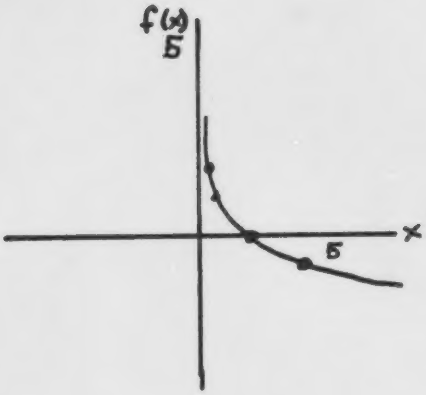
22. $f(x) = \log_5 x$ $x = 5^{f(x)}$

x	0.2	1	5	11.2
f(x)	-1	0	1	1.5



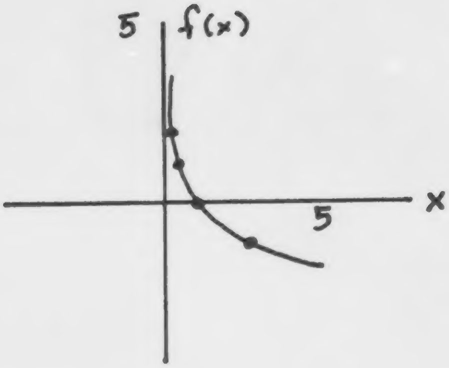
23. $f(x) = \log_{0.3} x$ $\qquad x = 0.3^{f(x)}$

x	3.33	1	0.3	0.09
f(x)	-1	0	1	2



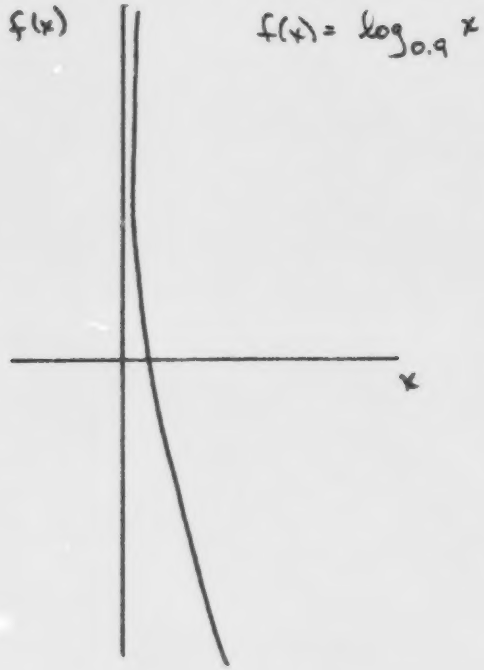
24. $f(x) = \log_{0.4} x$ $\qquad x = 0.4^{f(x)}$

x	2.5	1	0.4	0.16
f(x)	-1	0	1	2



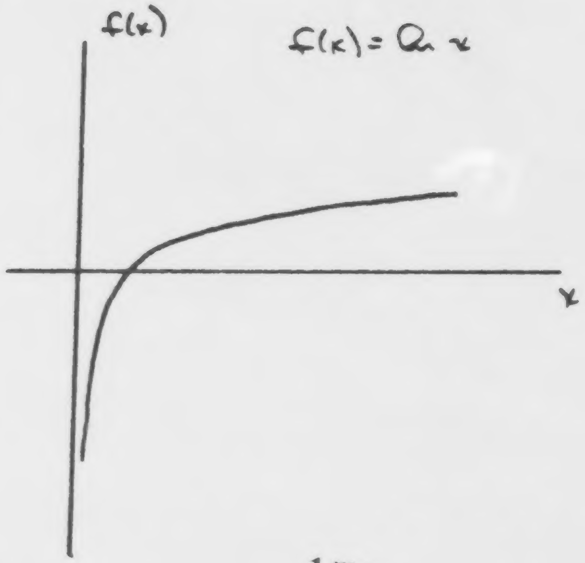
25. $f(x) = \log_{0.9} x \qquad x = 0.9^{f(x)}$

x	1.11	1	0.9	0.81
$f(x)$	-1	0	1	2



26. $f(x) = \ln x = \log_e x \qquad x = e^{f(x)}$

x	0.37	1	2.72	7.39	20.09
$f(x)$	-1	0	1	2	3



27. $f(x) = \log 5 = 0.69897$

28. $f(x) = \log 17 = 1.23045$

29. $f(x) = \log 0.003 = -3.47712$

30. $f(x) = \ln 10 = 2.30258$

31. $f(x) = \ln 3/5 = -0.51083$

32. $f(x) = \log 3/5 = -1.77815$

33. $f(x) = \ln 0.3 = -1.2040$

34. $f(x) = 4 \log 20 = 4(1.30103) = 5.20412$

35. $f(x) = -3 \ln 2 = -2.07944$

36. $117(1.127)^x = 150(1.06)^x$

$$\frac{1.127^x}{1.06^x} = \frac{150}{117}$$

$$\left(\frac{1.127}{1.06}\right)^x = 1.28205$$

$$x = \frac{\ln 1.28205}{\ln 1.06321} = 4.054 \qquad \simeq \text{ 4 years and 6 months}$$

37. $$\frac{55(1.086)^x}{1600(1.02)^x} = 0.12$$

$$\left(\frac{1.086}{1.02}\right)^x = 0.12\left(\frac{1600}{55}\right)$$

$$1.0647^x = 3.4909$$

$$x = \frac{\ln 3.4909}{\ln 1.0647} = 19.94 \qquad x = \text{approximately 20 years}$$

38. $150{,}000 = 75{,}000R^{12}$

$R^{12} = 2$

$12 \ln R = 1.2$

$\ln R = \ln 2/12 = 0.05776$

$\log_e R = 0.05776$

$R = e^{0.05776}$ average rate = 5.78%

1980 $75{,}000(e^{0.05776})^2 = \$84{,}184.27$

1982 $75{,}000(e^{0.05776})^4 = \$94{,}493.22$

[Obviously, not a good investment!]

39. $361R^{25} = 2208$

$R^{25} = 6.11634$

$25 \ln R = \ln 6.11634$

$\ln R = 0.07244$

$\log_e R = 0.07244$

$R = e^{0.07244}$

$f(x) = 361(e^{0.07244})^{30} = \$3,171.87$

CHAPTER 8

MATHEMATICS OF INVESTMENT AND FINANCE

The concepts in this chapter should be required learning for all adults. Basics such as future value, present value, nominal and effective rates, annuities, amortization, repayment rates, and perpetuities are discussed in an applied manner.

Once these topics are understood, our business students will begin to comprehend the complex buisness world.

Summaries of formulas (see pages 324 and 339) pull together the material to aid learning.

The sections on amortization and repayment rate will enable the student to speak intelligently with a realtor. The section on annuities will be invaluable to the student desiring to obtain a loan for "that" car.

Comprehensive tables are provided in the appendices.

Exercise 8.4, pp. 326-327.

1. $Pn = 100(1 + .07)^5 = 140.26$
2. $Pn = 500[1 + (.06/12)]^{12(4)} = 635.24$
3. $Pn = 1000e^{.05(10)} = 1648.72$
4. $Pn = 50(1 + .055)^2 = 55.65$
5. $Pn = 250(1 + .12/12)^{12(3)} = 357.69$
6. $Pn = 300(1 + .18/12)^{12(1)} = 358.69$
7. $Pn = 2000e^{.075(5)} = 2909.98$

   $2000 = 4000e^{-.075(n)}$ $n = 9.242$

8. $100 = 121.55(1 + .05)^{-n}$ $n = 4$
9. $500 = 734.66(1 + i)^{-5}$ $i = .08$
10. $P = 140.26(1 + .07)^{-5} = 100$
11. $P = 346.95(1 + .06/4)^{-4(6)} = 242.71$
12. $P = 2087.90(1 + .18/12)^{-12(8)} = 500$
13. $P = 973.75(1 + .05)^{-2} = 883.22$
14. $P = 181.67(1 + .12/12)^{-12(5)} = 100$
15. $P = 200.97(1 + .07/12)^{-12(10)} = 100$

16.

Month	Money	
January	$500\ (1 + .06/12)^{12(12/12)}$	= 530.84
March	$400\ (1 + .06/12)^{12(10/12)}$	= 420.46
April	$100\ (1 + .06)/12)^{12(9/12)}$	= 104.59
May	$600\ (1 + .06/12)^{12(8/12)}$	= 624.42
July	$700\ (1 + .06/12)^{12(6/12)}$	= 721.26
September	$200\ (1 + .06/12)^{12(4/12)}$	= 204.03
November	$500\ (1 + .06/12)^{12(2/12)}$	= 505.01
	3000	3110.61

Interest earned = \$3110.61 - \$3000 = \$110.61

17.

Month	Money	
February	$250\ (1 + .06/12)^{-12(1/12)}$	= 248.76
April	$850\ (1 + .06/12)^{-12(3/12)}$	= 837.38
June	$255\ (1 + .06/12)^{-12(5/12)}$	= 248.72
November	$295\ (1 + .06/12)^{-12(10/12)}$	= 280.65
December	$720\ (1 + .06/12)^{-12(11/12)}$	= 681.56
		2297.07

Exercise 8.5, p. 332.

Nominal → compounded continuously

Effective → compounded annually

1. $(1 + i/1)^1 = (1 + .05/4)^4$

$1 + i = 1.050945$

$i = 0.050945 \quad \simeq 5.09\%$

2. $(1 + i/1)^1 = (1 + .075/4)^4$

$1 + i = 1.077136$

$i = 0.\text{-}77136 \quad \simeq 7.71\%$

$(1 + i/1)^1 = (1 + .07/12)^{12}$ 7% quarterly is better investment.

$1 + i = 1.07229$

$i = 0.07229 \quad \simeq 7.23\%$

3. $i_{nom} = \ln(1 + i_{eff})$

$= \ln(1 + 0.0806)$

$= 0.0775$ $\simeq 7.75\%$

4. $i_{nom} = \ln(1 + i_{eff})$

$= \ln(1 + 0.0592)$

$= 0.0575$ $\simeq 5.75\%$

5. $i_{eff} = e^{nom} - 1$

$= e^{0.12} - 1$

$= 0.127497$ $\simeq 12.75\%$

6. $10{,}049 = 5000(1 + i)^9$

$(1 + i)^9 = 2.0098$

$1 + i = 1.080646$

$i = 0.080646$ $\simeq 8.06\%$

7. $(1 + i/12)^{12} = (1 + .06/365)^{365}$

$1 + i/12 = (1 + .06/365)^{365/12}$

$1 + i/12 = 1.005012$

$i/12 = 0.005012$

$i = 0.060145$ $\simeq 6.01\%$

Exercise 8.6, pp. 339-340.

1. $A_n = 200 \left(\frac{e^{.065(5)} - 1}{.06/12}\right)$

$= \$13{,}994.35$

2. $(1 + i/1)^1 = (1 + .05/12)^{12}$

$1 + i = 1.05116$

$i = 0.05116$ $\simeq 5.12\%$ (annual rate)

$A_n = 100 \left(\frac{(1 + .0512/4)^{4(8)} - 1}{.0512/4}\right)$

$= \$3{,}924.26$

3. $1000 = P\left[\dfrac{1 - (1 + .12/12)^{-12(1)}}{.12/12}\right]$

$1000 = P(11.25507747$

$P = \$88.85$

4. $2500 = P\left[\dfrac{1 - e^{-(.12)(3)}}{.12/4}\right]$

$2500 = P(10.0774558)$

$P = \$248.08$

5. $(1 + i/1)^1 = (1 + .08/4)^4$

$1 + i = 1.082432$

$i = 0.082432 \qquad \approx 8.24\%$

$3000 = P\left[\dfrac{1 - (1 + .0824/12)^{-(12)(3)}}{.0824/12}\right]$

$3000 = P(31.79933327)$

$P = \$94.34$

6. $(1 + i/2)^2 = (1 + .05/1)^1$

$(1 + i/2) = (1 + .05)^{1/2}$

$1 + i/2 = 1.024695$

$i = 0.04939$

$A = 500\left(\dfrac{(1 + .049/1)^{1(10)} - 1}{.049/1}\right)$

$= 500\ (12.51934003)$

$= \$6259.67$

7. $10{,}000 = P\left(\dfrac{1 - (1 + .10/1)^{-(1)(5)}}{.10}\right)$

$10{,}000 = P(3.790786769)$

$p = \$2637.97$

8. $$ECP = 30{,}000 + 200\left(\frac{1 - (1 + .12/12)^{-12(20)}}{.12/12}\right)$$

$$= 30{,}000 + 200(90.8194)$$

$$= \$48{,}163.88$$

9. This problem is a lead in for the section on amortization.

Month	Payment	Interest Paid	Principal Repaid	Principal Outstanding
				$63,000.00
September	$564.54	$538.13	$26.41	62,973.59
October	564.54	537.90	26.64	62,946.95
November	564.54	537.67	26.87	62.920.08
December	564.54	537.44	27.10	62.892.98
		$2151.14		

10. (a) $$550 = 20\left(\frac{(1 + .06/12)^{12(n/12)} - 1}{.06/12}\right)$$

$$2.75/20 = (1.005)^{n} - 1$$

$$1.005 = 22.75/20$$

$$= 1.1375$$

$$n(\ln 1.005) = \ln 1.1375$$

$$n = 25.8 \quad \approx 26 \text{ months}$$

Two years or 24 payments made--thus 26 - 24, two more months to go.

26 × $20 = $520.00 (approximately)

(b) on time

$$550 = p\left(\frac{1 - (1 + .18/12)^{-24}}{.18/12}\right)$$

P = $27.46

$27.46 × 24 = $659.04

Difference: $659.04 - 520 = $139.04

11. Need: $P_n - 10,000\ (1 + .1)^{10} = \$25,937.42$

Thus: $25,937.42 = P\left[\dfrac{(1 + .06/12)^{120} - 1}{.06/12}\right]$

$P = \$158.27$

12. $A_n = 52.17\left[\dfrac{(1 + .05/12)^{10} - 1}{.05/12}\right]$

$= \$531.59$

Exercise 8.7, pp. 345-346.

1. (a) $15,000 = P\left[\dfrac{1 - (1 + .10/12)^{-12(3)}}{.10/12}\right]$

$P = 484.01$

(b)

Month	Repay	Interest	Principal Repaid	Principal Outstanding
0				\$15,000.00
1	\$484.01	\$125.00	\$359.01	14,640.99
2	484.01	122.01	362.00	14,278.99
3	484.01	118.99	365.02	13,913.97
4	484.01	115.95	368.06	13,545.91
5	484.01	112.88	371.13	13,714.78
6	484.01	109.79	374.22	12,800.56
			2199.44	

2. $9000 = P\left[\dfrac{1 - (1 + .12/12)^{-120}}{.12/12}\right]$

$P = \$129.12$

Amount owed $= 129.12\left[\dfrac{1 - (1 + .01)^{-111}}{.01}\right]$

$= \$8,633.21$

Total amount owed = \$129.12 + 8633.21 = \$8762.33

3.

Month	Payment	Principal left	Interest Owed	Principal paid
0	----	$300.00	----	----
1	$14.98	289.52	$4.50	$10.48
2	14.98	278.88	4.34	10.64
3	14.98	268.08	4.18	10.80
4	14.98	257.12	4.02	10.96
5	14.98	246.00	3.86	11.12
6	14.98	234.71	3.69	11.29
7	14.98	232.25	3.52	11.46
8	14.98	211.62	3.35	11.63
9	14.98	199.81	3.17	11.81
10	14.98	187.83	3.00	11.98
11	14.98	175.67	2.82	12.16
12	14.98	163.33	2.64	12.34
			$43.09	

4.

Month	Payment	Principal left	Interest	Principal paid
0		$3200.00		
1	$150.64	3081.36	$32.00	$118.64
2	150.64	2961.53	30.81	119.83
3	150.64	2840.51	29.62	121.02
4	150.64	2718.28	28.41	122.23
5	150.64	2594.82	27.18	123.46
6	150.64	2740.13	25.95	124.69
7	150.64	2344.19	24.70	125.94
8	150.64	2216.99	23.44	127.20
9	150.64	2088.52	22.17	128.47
10	150.64	1958.77	20.89	129.75
11	150.64	1827.72	19.59	131.05
12	150.64	1695.36	18.28	132.36

13	$150.64	$1561.67	$16.95	$133.69
14	150.64	1426.65	15.62	135.02

[Check: $150.64 \frac{1 - (1.01)^{-10}}{.01} = \$1426.76$]

(b) $2800 - 1426.64 = $1373.35

(c) (1373.35)(.80) = $1098.68

(d) $1098.68 = P \frac{1 - (1 + .14/12)^{-12}}{.14/12}$

$P = \$98.65/\text{month}$

Exercise 8.8, p. 349.

For problems 1-7, use $f = \frac{1 - (1 + i/k)^{-in}}{i/k}$

1. $f = \frac{1 - (1 + .10/12)^{-12(4)}}{.10/12} = 39.42816$ R = 25.36/$1000

2. $f = \frac{1 - (1 + .14/12)^{-12(4)}}{.14/12} = 36.5945\ldots$ R = 27.33/$1000

3. $f = \frac{1 - (1 + .10/12)^{12(3)}}{.10/12} = 30.9912$ R = 32.27/$1000

4. $f = \frac{1 - (1 + .14/12)^{-12(3)}}{.10/12} = 29.2589\ldots$ R = 34.18/$1000

5. $f = \frac{1 - (1 + .10/12)^{-12(30)}}{.10/12} = 113.950082$ R = 8.78/$1000

6. $f = \frac{1 - (1 + .10/12)^{-12(20)}}{.10/12} = 103.6246\ldots$ R = 9.65/$1000

7. $f = \frac{1 - (1 + .10/12)^{-12(25)}}{.10/12} = 110.0472\ldots$ R = 9.09/$1000

8. $f = \left(\frac{1 - (1 + .135/12)^{-12(4)}}{.135/12}\right) = 36.93263674$

$R = P/A \times 1000$

$27.08 = 115/A \times 1000$

$A = \$4,246.68$

9. $R = \frac{564.54}{63,000} \times 1000 = 8.96/\$1000$

10. $i = 10\%$

(a) $P = 750 - 1800/12 = 600$

$.8A = 600 \left(\frac{1 - (1 + .10/12)^{-12(30)}}{.10/12}\right)$

$.8A = 600(113.95082)$

$A = \$85,463.11$

(b) $.8(85,463.11) + 30,000 = \$98,370.49$

Exercise 8.9, p. 351.

1. $A_P = 3000/0.075 = \$40,000$

2. $500,000 = \frac{P}{(.0775)/52}$

$P = \$745.19$

3. $(1 + i/1)^1 = (1 + .08/365)^{365}$

$1 + i = 1.083277$

$i = 0.083277 \quad \approx 8.33\%$

$100,000 = P/(.0833/12)$

$P = \$694.17$

4. $Ap = 100/.05 = \$2000$

190

5. (a) $(1 + i/1)^1 = (1 + .04/2)^2$

$1 + i = 1.0404$

$i = 1.0404$

$i = 4.04\%$

(b) 8%

(c) $1 + i = (1 + .075/52)^{52}$

$1 + i = 1.0778$

$i = 7.78\%$

CHAPTER 9

RULES OF DIFFERENTIAL CALCULUS

The basic concepts of limits and continuity are discussed prior to the presentation of the elementary rules of differential calculus. A summary of rules appears on pages 401-403. "Rule to use" table addresses when base is a constant or a function, depending how the base is raised, the student can determined which rule applies.

Sixty-five examples aid the student in learning the concepts. The Study Guide complements Chapter 9.

Exercise 9.3, pp. 369-371.

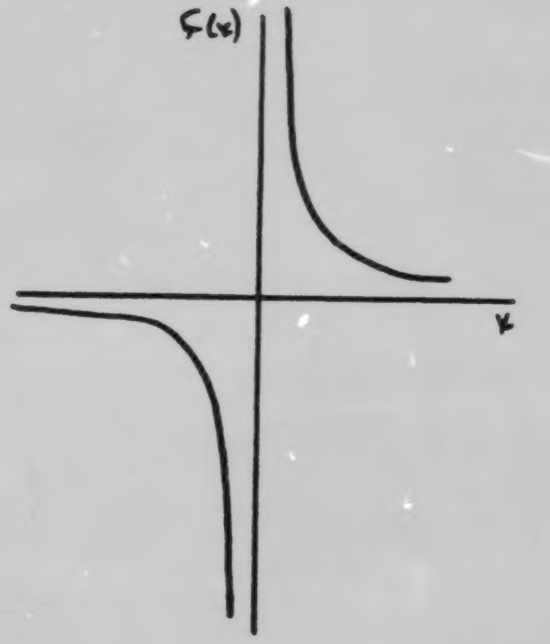
1. 24
2. -1
3. 1
4. 6
5. 2
6. 2.71828...
7. 1
8. ∞
9. 13
10. 0
11. 1
12. 0
13. 1
14. 1.60943
15. 0.2
16. 1
17. $-\infty$
18. 13
19. 0
20. undefined
21. 0.5
22. 0
23. -1
24. 4
25. undefined
26. undefined
27. -4
28. 1
29. 0
30. 0.43429

31. $f(x) = \frac{1}{x^3}$

$\lim x \to 0$

x	-2	-1	-1/2	0	1/2	1	2
f(x)	.125	-1	-8	*	8	1	.125

*undefined

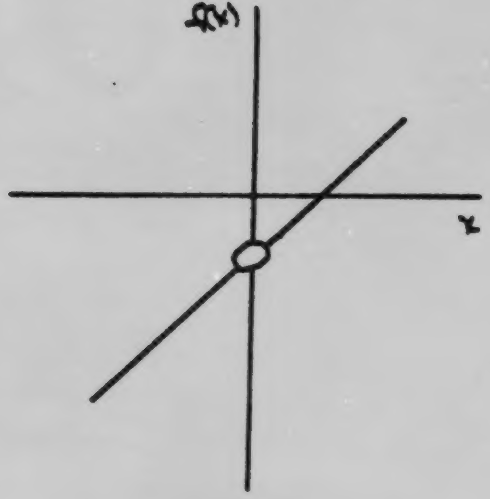


32. $f(x) = \frac{x^2 - x}{x}$

$\lim x \to 0$

x	-2	-1	-1/2	0	1/2	1	2
f(x)	-3	-2	-3/2	*	-1/2	0	1

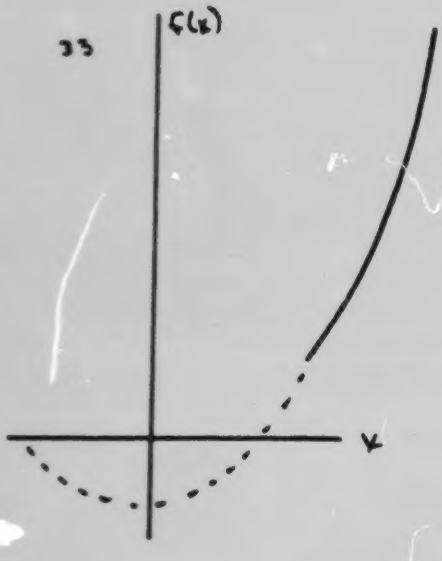
*undefined



33. $f(x) = -10 + x^2$

lim $x \to 5+$

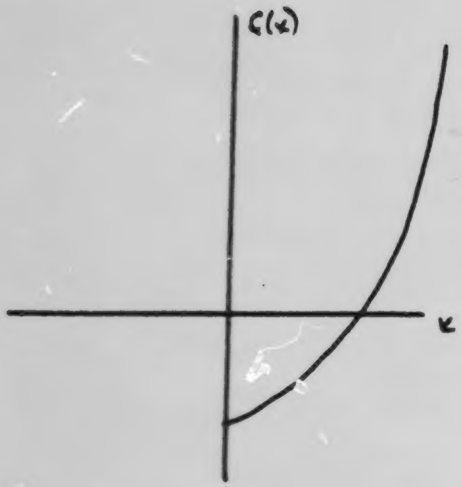
x	10	8	6	5.5	5.2	5.1	5
f(x)	90	54	26	20.25	17.04	15.01	15



34. $f(x) = -10 + x^2$

lim $x \to 5-$

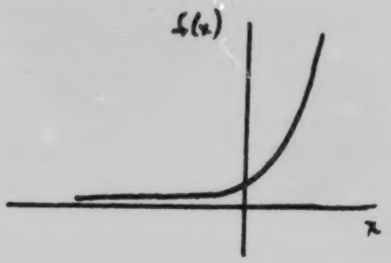
x	0	1	2	3	4	4.5	4.7	4.9	5
f(x)	-10	-9	-6	-1	6	10.25	12.09	14.01	15



35. $f(x) = e^x$

$\lim x \to -\infty +$

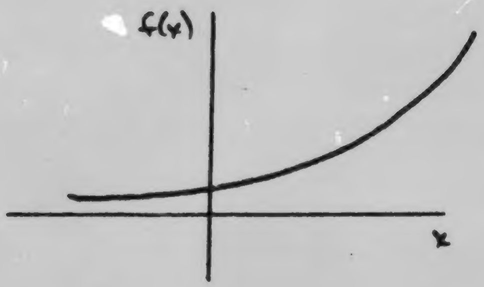
x	1	0	-1	-2	-10	-100	-1000
f(x)	2.72	1	.37	.14	4.5×10^{-5}	3.72×10^{-44}	$-\infty$



36. $f(x) = 3^x$

$\lim x \to 2-$

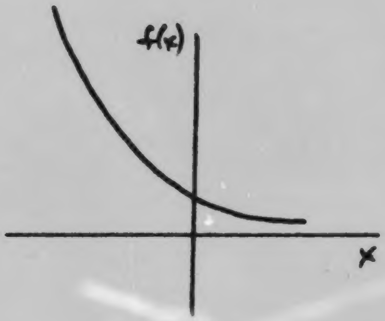
x	-1	0	1	1.5	1.7	1.9	2
f(x)	.33	1	3	5.20	6.47	8.06	9



37. $f(x) = 3^{-x}$

$\lim x \to -2+$

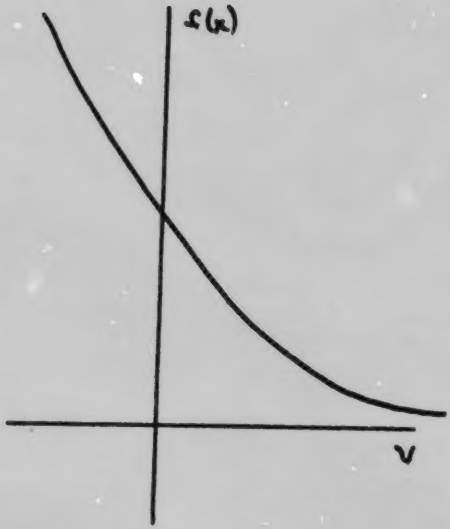
x	-2	-1.9	-1.7	-1.5	-1	0	+1
f(x)	9	8.06	6.47	5.20	3	1	.33



38. $f(x) = 0.2^{x}$

$\lim x \to 2-$

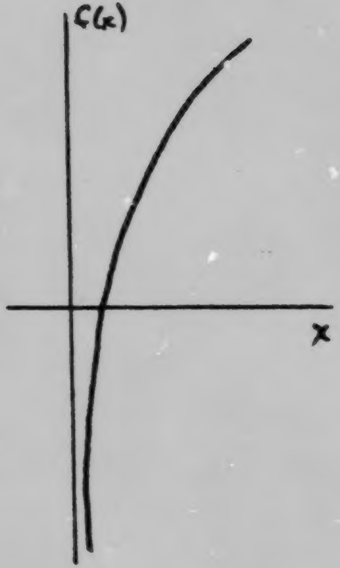
x	-1	0	1	1.5	1.7	1.9	2
f(x)	5	1	0.2	0.089	0.065	0.047	0.04



39. $f(x) = e^{-x}$

$\lim x \to \infty$

x	0	1	10	100	1000	10,000	∞
f(x)	1	.37	4.5E-5	3.72E-44	-	-	-

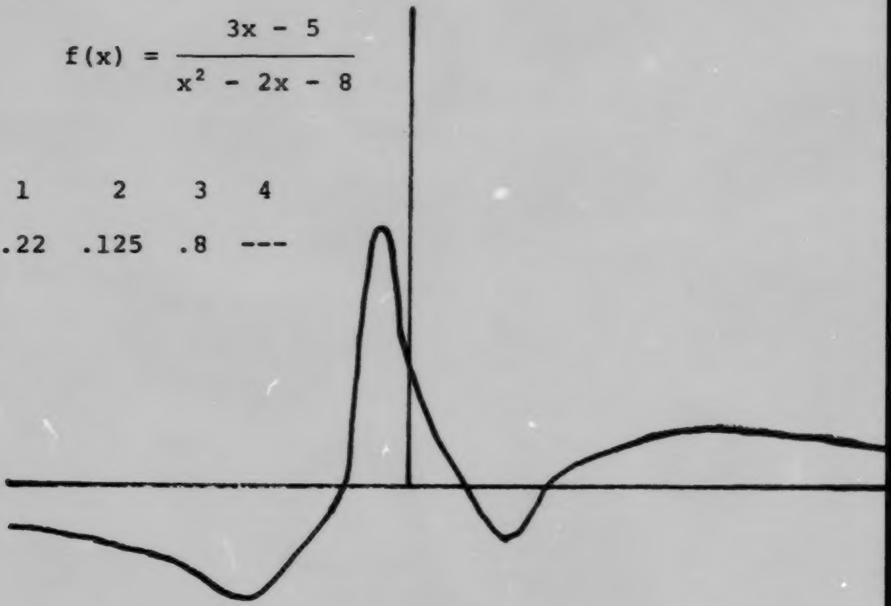


40. See example 17 on pp. 3-8-309.

41. for $+\infty$, $f(x) \to 0+$ $\qquad f(x) = \dfrac{3x - 5}{x^2 - 2x - 8}$

for $-\infty$, $f(x) \to 0-$

x	-3	-2	-1	0	1	2	3	4
f(x)	-2	--	1.6	.625	.22	.125	.8	---



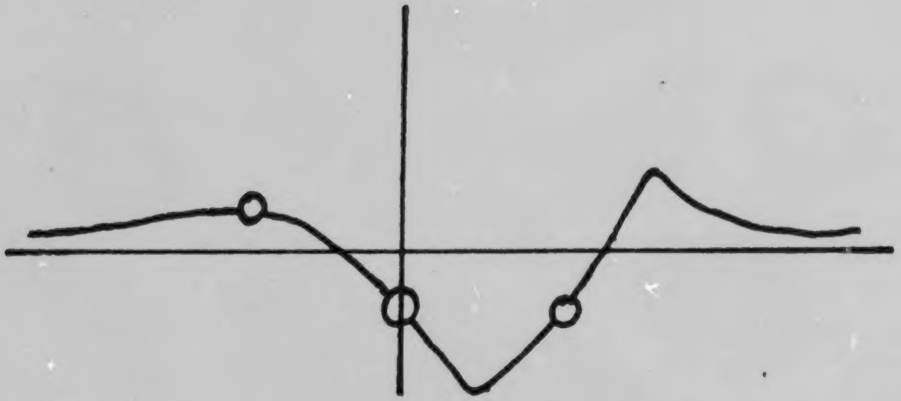
42. for $+\infty$, $f(x) \to 0+$

for $-\infty$, $f(x) \to 0+$

$$f(x) = \frac{x + 1}{x^3 - 4x}$$

x	-4	-3	-2	-1	0	1	2	3
f(x)	.04	.15	*	0	*	-.67	*	.27

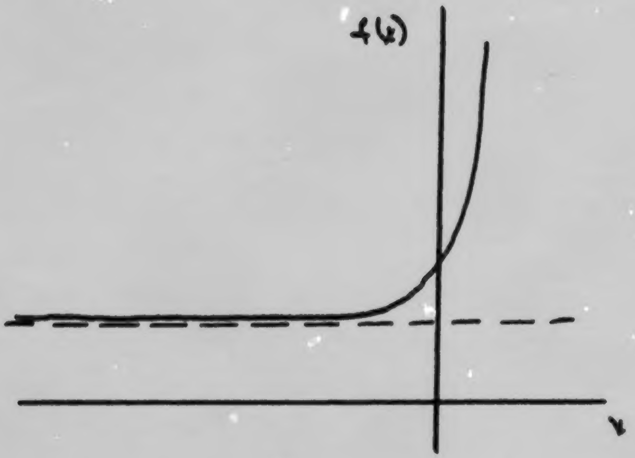
*undefined



43. $f(x) = e^x + 2$

for $+\infty$, $+\infty$

$-\infty$, $+2$

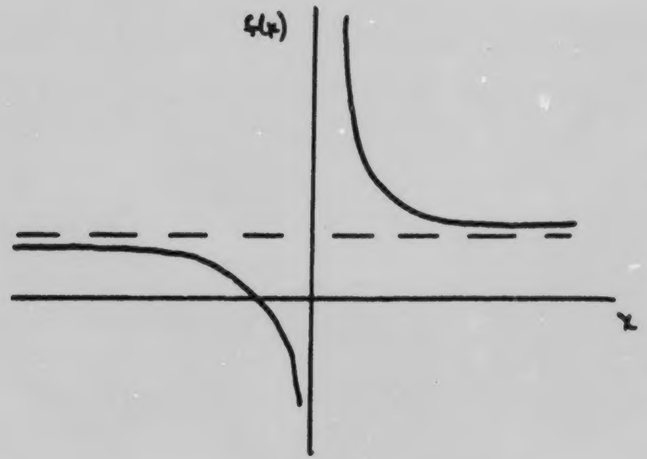


44. $f(x) = \frac{x + 1}{x}$ for $+\infty$, $+1$

for $-\infty$, $+1$

x	-4	-3	-2	-1	0	1	2	3	4
f(x)	.75	.67	.5	0	*	2	1.5	1.33	1.25

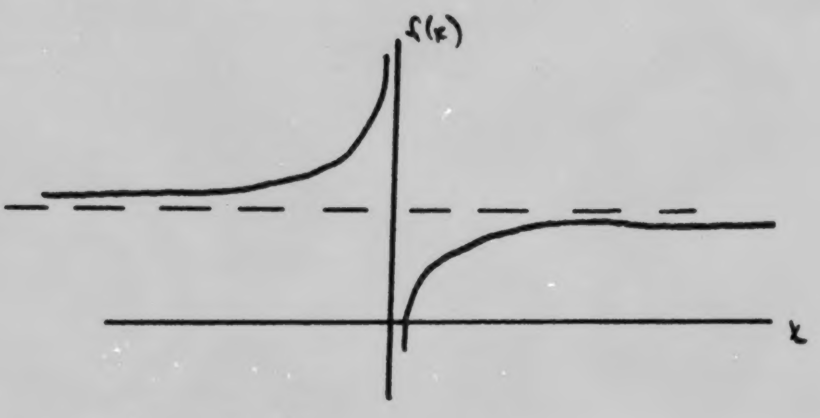
*undefined



45. $f(x) = \frac{1}{1 + 2^{1/x}}$ for $+\infty$, 0.5

for $-\infty$, 0.5

x	-3	-2	-1	0	1	2	3
f(x)	.55	.59	.67	*	.33	.41	.44



46. $x = 0$; missing point

47. $x = +2, -1$; missing point

48. $x = 1$; missing point

49. $x = -5$; removable discontinuity

50. $x = 0$; missing point

51. $x = 0, +2, -2$; missing point

52. none

53. $x = 0$; missing point; $x < x \leq 3-$ jump discontinuity

54. $x = \pm 3$; missing point

55. $x = 1.0986$; missing point

Exercise 9.4, pp. 380-381.

1. $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{[(x + \Delta x) + 2] - [x + 2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x}{x}$$

$$= 1$$

2. $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 + (x + \Delta x) - 3] - [x^2 + x - 3]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta y + (\Delta x)^2 + \Delta x}{\Delta X}$$

$$= \lim_{\Delta x \to 0} 2x + \Delta x + 1$$

$$= 2x + 1$$

3. $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{[3(x + \Delta x)^2 + 5(x + \Delta x) + 7] - [3x^2 + 5x + 7]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{6x\Delta x + 3(\Delta x)^2 + 5\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 6x + 3\Delta x + 5$$

$$= 6x + 5$$

4. $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^3 + 2(x + \Delta x)^2 + (x + \Delta x) - 8] - [x^3 + 2x^2 + x - 8]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 4x\Delta x + 2(\Delta x)^2 + \Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 3x^2 + 3x\Delta x + (\Delta x)^2 + 4x + 2\Delta x + 1$$

$$= 3x^2 + 4x + 1$$

5. $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{[2(x + \Delta x)^3 - 2] - [2x^3 - 2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{6x^2\Delta x + 6x(\Delta x)^2 + 2(\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 6x^2 + 6x\Delta x + 2\Delta x$$

$$= 6x^2$$

6. $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{[9(x + \Delta x)^2 - 4(x + \Delta x)] - [9x^2 - 4x]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{18x\Delta x + 9(\Delta x)^2 - 4\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 18x + 9\Delta x - 4$$

$$= 18x - 4$$

7. $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{[1/(x + \Delta x)^2] - [1/x^2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 - (x + \Delta x)^2}{(\Delta x)(x^2)(x + \Delta x)^2}$$

$$= \lim_{\Delta x \to 0} \frac{-2x - \Delta x}{(x^2)(x + \Delta x)^2}$$

$$= -2/x^3$$

8.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\frac{1}{[2(x + \Delta x) - 3]} - \frac{1}{[2x - 3]}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x - 3 - 2x - 2\ \Delta x + 3}{\Delta x[2(x + \Delta x) - 3][2x - 3]}$$

$$= \lim_{\Delta x \to 0} \frac{-2}{[2(x + \Delta x) - 3][2x - 3]}$$

$$= \frac{-2}{(2x - 3)^2}$$

9. $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\frac{1}{[(x + \Delta x)^2 + 4]} \quad \frac{1}{[x^2 + 4]^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[x^2 + 4]^2 \quad [(x + \Delta x)^2 + 4]^2}{\Delta x(x^2 + 4)^2[(x + \Delta x)^2 + 4]^2}$$

$$= \lim_{\Delta x \to 0} \frac{x^4 + 8x^2 + 16 - [x^2 + 2x\Delta x + (\Delta x)^2 + 4]^2}{\Delta x(x^2 + 4)^2[(x + \Delta x)^2 + 4]^2}$$

$$= \lim_{\Delta x \to 0} \frac{x^4+8x^2+16-x^4-4x^3\Delta x-6x^2\Delta x^2-4x\Delta x^3-\Delta x^4-16x\Delta x-8x^2-8\Delta x^2-16}{\Delta x(x^2 + 4)^2\ [(x + \Delta x)^2 + 4]^2}$$

$$= \lim_{\Delta x \to 0} \frac{-4x^3 - 6x^2\Delta x - 4x\Delta x^2 - \Delta x^3 - 16x}{(x^2 + 4)^2\ [(x + \Delta x)^2 + 4]^2}$$

$$= \frac{-4x(x^2 + 4)}{(x^2 + 4)^2\ (x^2 + 4^2)}$$

$$= \frac{-4x}{(x^2 + 4)^3}$$

10. $$f(x) = \frac{x^3 + x^2 + x}{x}$$

$$= x^2 + x + 1$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 + (x + \Delta x) + 1] - [x^2 + x + 1]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 + x + \Delta x + 1 - x^2 - x - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + \Delta x + 1$$
$$= 2x + 1$$

11. $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 + 5] - [x^2 + 5]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + \Delta x$$

$$= 2x$$

$$\frac{d^2y}{dx^2} = \lim_{\Delta x \to 0} \frac{2(x + \Delta x) - 2x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2$$
$$= 2$$

12. $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{[2(x + \Delta x)^2 + 5(x + \Delta x) + 7] - [2x^2 + 5x + 7]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 4x + 2\Delta x + 5$$

$$= 4x + 5$$

$$\frac{d^2y}{dx^2} = \lim_{\Delta x \to 0} \frac{[4(x + \Delta x) + 5] - [4x + 5]}{x}$$

$$= \lim_{\Delta x \to 0} 4$$

$$= 4$$

13. $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\frac{1}{(x + \Delta x)} - \frac{1}{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x)(x)}$$

$$= -1/x^2$$

$$\frac{d^2y}{dx^2} = \lim_{\Delta x \to 0} \frac{\frac{-1}{(x + \Delta x)^2} - \frac{-1}{x^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-x^2 + x^2 + 2x\Delta x + \Delta x^2}{\Delta x(x + \Delta x)^2 \ (x^2)}$$

$$= \lim_{\Delta x \to 0} \frac{2x + \Delta x}{(x + \Delta x)^2(x^2)}$$

$$= 2/x^3$$

14. $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\frac{1}{[(x + \Delta x)^2 - 5]} \quad \frac{1}{[x^2 - 5]}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 - 5 - x^2 - 2x\Delta x - \Delta x^2 + 5}{\Delta x[(x + \Delta x)^2 - 5][x^2 - 5]}$$

$$= \lim_{\Delta x \to 0} \frac{-2x}{[(x + \Delta x)^2 - 5][x^2 - 5]}$$

$$= -2x/(x^2 - 5)^2$$

$$\frac{d^2y}{dx} = \lim_{\Delta x \to 0} \frac{\frac{-2(x + \Delta x)}{[(x+\Delta x)^2 - 5]^2} - \frac{-2x}{[x^2 - 5]^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2(x + \Delta x)(x^2 - 5)^2 + 2x[(x + \Delta x)^2 - 5]^2}{\Delta x[(x + \Delta x)^2 - 5]^2 \ (x^2 - 5)^2}$$

$$= 8x^2/(x^2 - 5)^3$$

15. $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\frac{3}{(x + \Delta x)^2} - \frac{3}{x^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-6x\Delta x - 3\Delta x^2}{\Delta x(x + \Delta x)^2(x^2)}$$

$$= \lim_{\Delta x \to 0} \frac{-6x - 3\Delta x}{(x + \Delta x)^2(x^2)}$$

$$= -6/x^3$$

$$\frac{d^2y}{dx^2} = \lim_{\Delta x \to 0} \frac{\frac{-6}{(x + \Delta x)^3} - \frac{-6}{x^3}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{18x^2\Delta x + 18x\Delta x^2 + 6\Delta x^3}{\Delta x(x + \Delta x)^3x^3}$$

$$= \lim_{\Delta x \to 0} \frac{18x^2 + 18x\Delta x + 6\Delta x^2}{(x + \Delta x)^3(x^3)}$$

$$= 18/x^4$$

Exercise 9.5, pp. 403-406.

1. $f'(x) = 0$
2. $f'(x) = 0$
3. $f'(x) = 0$
4. $f'(x) = 0$
5. $f'(x) = 0$
6. $f'(x) = 2x$
7. $f'(x) = 9x^8$
8. $f'(x) = -4x^{-5}$
9. $f'(x) = 1/3x^{-2/3}$
10. $f'(x) = -2/5x^{-7/5}$
11. $f'(x) = 9/4x^{5/4}$
12. $f'(x)\ -4/5x^{-9/5}$
13. $f'(x) = 0.3x^{-0.7}$
14. $f'(x) = -0.25x^{-1.25}$
15. $f'(x) = ex^{e-1}$
16. $f'(x) = 6x^2$
17. $f'(x) = -20x^{-6}$
18. $f'(x) = 9x^{-2/3}$
19. $f'(x) = -4x^{-9/5}$
20. $f'(x) = 3x^{-5/2}$
21. $f'(x) = -2/5x^{-11/5}$
22. $f'(x) =$
23. $f'(x) = 22/7ex^{e-1}$
24. $f'(x) = -4/15x^{-5/9}$
25. $f'(x) = 2$
26. $f'(x) = -4$

27. $f'(x) = 2/5$

28. $f'(x) = -2 + 2x$

29. $f'(x) = 3x^2 - 6x - 1$

30. $f'(x) = 3x^2 + 20x + 31$

31. $f'(x) = 2/5x - x + x^2 - 4x^3$

32. $f'(x) = 2(x^2 + 3) + (3 + 2x)(2x)$

33. $f'(x) = (-1 + 4x)(x + x^3) + (1 - x + 2x^2)(1 + 3x^2)$

34. $f'(x) = (3x^2 + 4x^3)(2x - x^2) + (x^3 + x^4)(2 - 2x)$

35. $f'(x) = (1/2x^{-1/2} - 1)(4x + 5x^2) + (x^{1/2} - x)(10x)$

36. $f'(x) = -1(2 + x^2)(3 + x^3) + (1 - x)(2x)(3 + x^3)$

$+ (1 - x)(2 + x^2)(3x^2)$

37. $f'(x) = (4x)(7x^3)(x - x^2) + (4 + 2x^2)(21x^2)(x - x^2)$

$+ (4 + 2x^2)(7x^3)(1 - 2x)$

38. $f'(x) = \dfrac{2x(2x) - 2x(2x)}{(x^2 + 5)} = 0$

39. $f'(x) = \dfrac{2(4 + 3x) - 3(2x + 1)}{(4 + 3x)^2}$

40. $f'(x) = \dfrac{4x(x^2 + x - 3) - (2x + 1)(5 + 2x^2)}{(5 + 2x^2)^2}$

41. $f'(x) = \dfrac{-(2x)(1)}{(-3x + x^2)^2}$

42. $f'(x) = \dfrac{-1(2x + 5)}{(x^2 + 5x + 1)^2}$

43. $f'(x) = \dfrac{(2x)(1 + x + x^2) - (1 + 2x)(x^2)}{(1 + x + x^2)^2}$

44. $f'(x) = 2(1 + x^2)(2x)$

45. $f'(x) = 3(3x^2 - 2x + 4)^2(6x - 2)$

46. $f'(x) = 10(1 + 5x - x^2)^9(5 - 2x)$

47. $f'(x) = 1/2(x^2 + 1)^{-1/2}(2x)$

48. $f'(x) = 1/3(x^2 + 1)^{-2/3}(2x)$

49. $f'(x) = 3(9 + 10x^2 + 5x^3)^2 \ (20x + 15x^2)$

50. $f'(x) = e(1 + x)^{e-1}$

51. $f'(x) = (1)(2^x)\ln 2$

52. $f'(x) = (2x + 16x^2)3^{x^2 + 4x^4} \ln 3$

53. $f'(x) = (1)(e^x) = e^x$

54. $f'(x) = 2xe^{7} + x^2$

55. $f'(x) = 1(e^x)(e^{e^x})$

56. $f'(x) = (2x + 3x^2)7^{x^2 + x^3} \ln 7$

57. $f'(x) = (-2x)(-3/5^{-2 - x})\ln -3/5 = 0$

58. $f'(x) = 0$

59. $f'(x) = 0$

60. $f'(x) = (1 + 6x + 3x^2)9^{x + 3x^2 + x^3} \ln 9$

61. $f'(x) = (-1/2 + 3x^2)(e^{-1/2x + x^3})$

62. $f'(x) = 2x/x^2 \ln 2$

63. $f'(x) = (2 - 2x)/(1 + 2x - x^2) \ln 9$

64. $f'(x) = -1/(3 - x) \ln 10$

65. $f'(x) = 1/(2 + x) \ln 1/2$

66. $f'(x) = (-1 + 8x)/(3 - x + 4x^2) \ln 10$

67. $f'(x) = -1/(3 - x)$

68. $f'(x) = (-1 + 2x)/(2 - x + x^2)$

69. $f'(x) = (-1 + 4x + 3x^2)/(3 - x + 2x^2 + x^3)$

70. $f'(x) = [2x/x^2] + [1/(3 + x)]$

71. $f'(x) = \dfrac{1 - 2x}{2 + x - x^2} + \dfrac{-1 + 2x}{-x + x^2}$

72. $f'(x) = [2x/(-3 + x^2)] + [x^3/8x^4]$

73. $f'(x) = \dfrac{3}{2 + 3x} - \dfrac{-2x}{1 - x^2}$

74. $f'(x) = \dfrac{1 + 4x}{3 + x + 2x^2} \quad \dfrac{6}{5 + 6x}$

75. $f'(x) = \dfrac{2(3 + x + 2x^2)(1 + 4x)}{(3 + x + 2x^2)^2}$

76. $f'(x) = (3/5)(-1 + 2x)/(2 - x + x^2)$

77. $f'(x) = (1)(e^x)/e^x = 1$

78. $f'(x) = \dfrac{3(2x)}{(4 + x^2)} \quad \dfrac{4(2x - 1)}{(x^2 - x + 3)}$

79. $f'(x) = \dfrac{2[3 - 5x(-5)]}{(3 - 5x)^2} + \dfrac{(1/2)(2 - x^3)^{-1/2}(-3x^2)}{(2 - x^3)^{1/2}}$

80. $f'(x) = \dfrac{4(14x)}{2 + 7x^2} - \dfrac{3(2x - 1)}{x^2 - x - 1}$

81. $\dfrac{dy}{dx} = (-2x - y)/(x - 2y)$

82. $\dfrac{dy}{dx} = (y + 3)/(-x + 2y + 2)$

83. $\dfrac{dy}{dx} = (-2x - y^3)/(-2y + 3xy^2 - 2)$

84. $\dfrac{dy}{dx} = (2x - 2y)/(3y^2 + 2x)$

85. $\dfrac{dy}{dx} = (2x - 1 + 2y - 2xy^2)/(2y - 1 - 2x + 2x^2y)$

86. $\dfrac{dy}{dx} = (2x - y)/(x - 2y)$

$\dfrac{dy}{dx} = (x - 2y)/(2x - y)$

87. $\frac{dy}{dx} = (-4x + y)/(3y^2 - x)$

$\frac{dy}{dx} = (3y^2 - x)/(-4x + y)$

88. $\frac{dy}{dx} = (y + 3)/(-x + 2y + 2)$

$\frac{dy}{dx} = (-x + 2y + 2)/(y + 3)$

89. $\frac{dy}{dx} = (2y - 3xy^2 + 2)/(2x + y^3)$

91. $$f'(x) = (x - 3x^2 + x^3)^{7 + 2x + 3x^2}\left[(2 + 6x)\ln(x - 3x^2 + x^3) + (7 + 2x + 3x^2)\left(\frac{1 - 6x + 3x^2}{x - 3x^2 + x^3}\right)\right]$$

92. $$f'(x) = (2 - x + x^2)^{x^2}\left[2x\ln(2 - x + x^2) + x^2\left(\frac{-1 + 2x}{2 - x + x^2}\right)\right]$$

93. $f'(x) = x^{x^2}\,[2x \ln x + x^2(1/x)]$

94. $f'(x) = (3 + x)^{e^x}\,[e^x \ln(3 + x) + e^x(\frac{1}{3 + x})]$

95. $f'(x) = (e^x)^{e^x}\,[e^x \ln e^x + e^x]$

96. $f'(x) = [2(3x - x^2)(3 - 2x)]\,(3 - x^2)^4 + (3x - x^2)^2[4(3 - x^2)^3(-2x)]$

97. $$f'(x) = [4(2 - x^2 + x^3)^3(-2x + 3x^2)](2 - 5x)^2 + (2 - x^2 + x^3)^4[2(2 - 5x)(-5)]$$

98. $$f'(x) = \frac{[4(1 + x^2)^3(2x)](3 - 2x)^2 - [2(3 - 2x)(-2)](1 + x^2)^4}{(1 + x^2)^8}$$

99. $$f'(x) = 2\left(\frac{x + 1}{x - 1}\right)\left(\frac{(x + 1) - (x - 1)}{(x - 1)^2}\right] = \frac{-4(x + 1)}{(x - 1)^3}$$

100. $f'(x) = 2x(3 + 4x^2)^3 + (\log_9 3 + x^2)[3(3 + 4x^2)^2(8x)]$

101. $$f'(x) = [2(7 - x)(-1)](2 + x)^x + (7 - x^2)[(2 + x)^x\{(2 + x) + x(\frac{1}{2 + x})\}]$$

102. $f'(x) = 4x + \frac{2(-2)}{(6 - 2x)\ln 10} - \frac{2x}{x^2}$

103. $f'(x) = 2[\ln(3 + x)(x^2 - 3x^4)] \left(\frac{1}{3 + x} + \frac{2x - 12x^3}{x - 3x^4}\right)$

104. $f'(x) = (-1)/(\log 3 - x)$

105. $f'(x) = [3(4 + x^2)^2(2x)](e^{x - x^2}) + (4 + x^2)^3[(1 - 2x)e^{x - x^2}]$

106. $f'(x) = [5^x\ln 5](x^e) + 5^x(ex^{e - 1}]$

107. $f'(x) = [2(2 - 3x^2)(-6x)](3^{1 + x}) + (2 - 3x^2)\ [1(3^{1 + x})\ln 3]$

108. $f'(x) = \left(\frac{-1}{4 - x}\right) e^{\ln(4 - x)}$

109. $f'(x) = 2xe^{2 \ln x} + x^2(2/x)e^{2 \ln x} = 4xe^{2 \ln x}$

110. $f'(x) = \frac{\left(\frac{1}{x \log 7}\right)(\log_8 e)}{(\log_7 x)} = \frac{0.24713}{x(\log_7 x)^2}$

111. $f'(x) = 3(1 + x)^2e^{2 - x^2} + (3x)[2(1 + x)1]e^{2 - x^2}$

$+ (3x)(1 + x)^2(-2x)e^{2 - x^2}$

112. $f'(x) = 5^x\ln x\left(\frac{4 - x}{x - x^2 + 4x^3}\right)^{1/2}$

$+ 5^x \left\{ 1/2 \left(\frac{4 - x}{x - x^2 + 4x^3}\right)^{-1/2} \frac{(1 - 2x + 12x^2) + (x - x^2 + 4x^3)}{(x - x^2 + 4x^3)^2} \right\}$

113. $f'(x) = \frac{2(-1)}{(5 - x)\ln 10} + 4xe^{x^2}$

114. $f'(x) = e^x\ln x + (e^x/x)$

115. $f'(x) = [-1(25^{2 - x})\ln 25] + 3^x\ln 3 - 3x^2$

116. $f'(x) = [(1 + x)/(2 - x)]^x$

$\times\{1(1 + x)/(2 - x) + x\left(\frac{-1(1 + x) - (2 - x)}{(2 - x)^2}\right)\}$

117. $f'(x) = [\ln 3 - x^2]^{2 + x^3}\left(3x^2 \ln(\ln 3 - x^2) + (2 + x^3)\left(\frac{-2x}{\ln 3 - x^2}\right)\right)$

118. $f'(x) = \frac{e^{-1/x}}{x^2} + 2ex^{e - 1} - 5^x \ln 5 + 4x^3$

119. $f'(x) = \frac{5^x \ln 5}{5^x \ln 7} = 0.82709$

120. $f'(x) = 2e^{2x}\left(\frac{3[2x]}{[x^2 + 5] \ln 10}\right)$

121. $f'(x) = 2 - 2x$

$f''(x) = -2$

122. $f'(x) = 2 + 2x$

$f'(x) = +2$

123. $f'(x) = \frac{(3 - x^2)(2x) - (x^2)(-2x)}{(3 - x^2)^2} = \frac{6x}{(3 - x^2)^2}$

$f''(x) = \frac{(3 - x^2)^2(6) - 6x[2(3 - x^2)(-2x)]}{(3 - x^2)^4}$

$= \frac{18 + 18x^2}{(3 - x^2)^3}$

124. $f'(x) = 3(2 - x + 3x^2)^2(-1 + 6x)$

$f''(x) = 6(2 - x + 3x^2)(-1 + 6x)^2 + 3(2 - x + 3x^2)^2(6)$

125. $f'(x) = -6 - 2x + 3x^2$

$f'(x) = -2 + 6x$

126. $f'(x) = 5^x \ln 5$

$f''(x) = 5^x \ln 5 = 5^x(2.59029$

127. $f'(x) = e^x$

$f''(x) = e^x$

128. $f'(x, = \frac{-1 + 6x}{2 - x + 3x^2}$

$f''(x) = \frac{6(2 - x + 3x^2) - (-1 + 6x)^2}{(2 - x + 3x^2)^2}$

129. $f'(x) = \frac{2(2x)}{3 + x^2} = \frac{4x}{3 + x^2}$

$$f''(x) = \frac{(3 + x^2)4 - 4x(2x)}{(3 + x^2)^2} = \frac{12 - 4x^2}{(3 + x^2)^2}$$

130. $f'(x) = \frac{12x^3 + 5}{3x^4 + 5x} + \frac{4x}{2x^2 + 7}$

$$f''(x) = \frac{(3x^4 + 5x)(36x^2) - (12x^3 + 5)^2}{(3x^4 + 5x)^2} + \frac{4(2x^2 + 7) - (4x)^2}{(2x^2 + 7)^2}$$

CHAPTER 10

DIFFERENTIAL CALCULUS--ADVANCED TOPICS AND APPLICATIONS

Applications of differential calculus presented in this chapter have several unique characteristics: (1) use of topics in preceeding chapters, (2) clarity of presentation, and (3) presentation by example.

Max-min is presented by a step by step approach (see page 410). Marginal analysis compares use of the first derivative and successive items as taught in economics courses. Table 10.3 displays the relationship generally only vaguely addressed.

Figure 10.5 displays the concept that maximum profit occurs for parabolic functions when marginal revenue equals marginal cost.

Elasticity is becoming more widely used as gasoline prices escalate. The example used discusses the price of coffee. Coffee prices rose dramatically in the late 70's.

In most production courses, the EOQ model is discussed when teaching inventory management. Students are exposed to this model as an application of differential calculus.

Exercise 10.1, pp. 419-422.

1. $f(x) = 2x^2 + 10x + 12$

$f'(x) = 4x + 10$

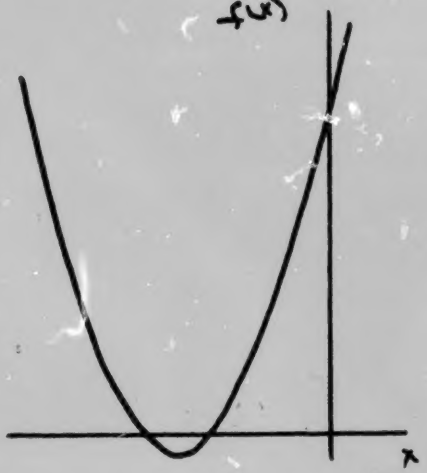
$0 = 4x + 10$

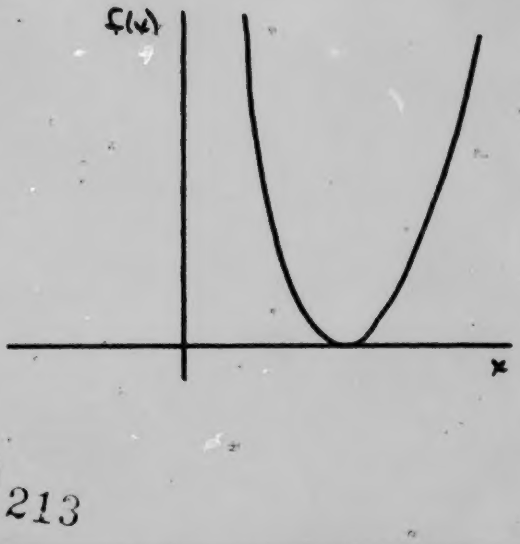
$x^* = -5/2$

$f(-5/2) = 25/2 - 25 + 12$

$= -1/2$

$f''(x) = +4$ (min)





2. $f(x) = x^2 - 10x + 25$

$f'(x) = 2x - 10$

$0 = 2x - 10$

$x^* = 5$

$f(5) = 25 - 50 + 25 = 0$

$f''(x) = +2$(min)

3. $f(x) = x^2 + 2x + 5$

$f'(x) = 2x + 1$

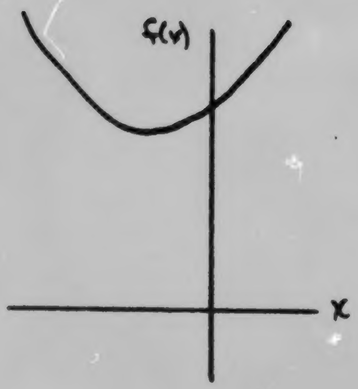
$0 = 2x + 1$

$x^* = -1/2$

$f(-1/2) = 1/4 - 1 + 5$

$= 4.25$

$f''(x) = +2$ (min)



4. $f(x) = -x^2 - 10x + 25$

$f'(x) = -2x - 10$

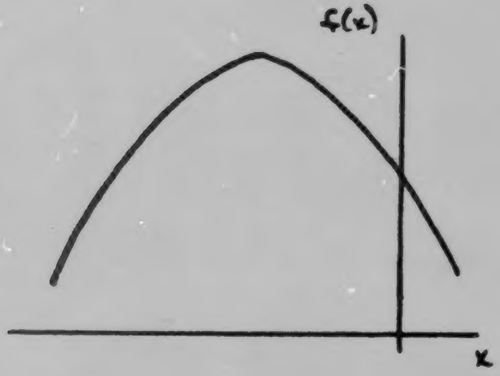
$0 = -2x - 10$

$x^* = -5$

$f(-5) = -25 + 50 + 25$

$= 50$

$f''(x) = -2$ (max)



5. $f(x) = -3x^2 + (2/3)x - 1/3$

$f'(x) = -6x + 2/3$

$0 = -6x + 2/3$

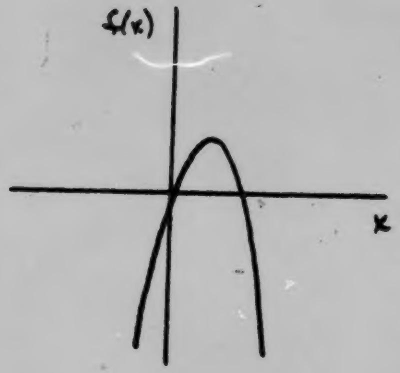
$x^* = 1/9$

$f(1/9) = -3(1/81) + 2/3(1/9) - 1/3$

$= -1/27 + 2/27 - 9/27$

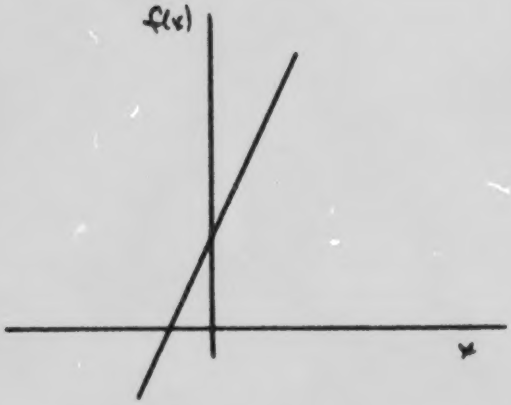
$= 8/27$

$f'(x) = -6$ (max)



6. $f(x) = 3 + 2x$

$f'(x) = 2$



7. $f(x) = x^3 + 10x^2 + 31x + 30$

$f'(x) = 3x^2 + 20x + 31$

$0 = 3x^2 + 20x + 31$

$x^* = \frac{-20 \pm \sqrt{20^2 - 4(3)(31)}}{2(3)}$

$= -4.22, -2.45$

$f(-4.22) = (-4.22)^3 + 10(-4.22)^2 + 31(-4.22) + 30$

$= 2.11$

$f(-2.45) = (-2.45)^3 + 10(-2.45)^2 + 31(-2.45) = 30$

$= -0.63$

$f''(x) = 6x + 20$

$f''(-4.22) = 6(-4.22) + 20$

$= -5.32$ (max)

$f''(-2.45) = 6(-2.45) + 20$

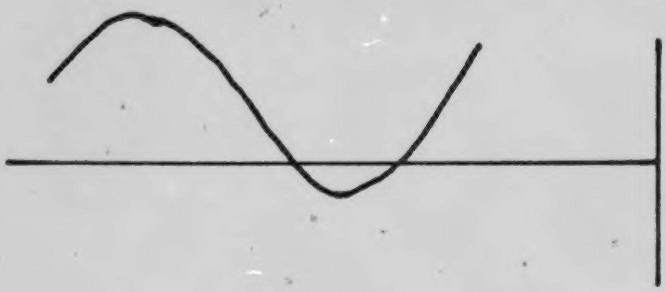
$= +5.3$ (min)

$0 = 6x + 20$

$x^{**} = -10/3 = -3.33$

$f(-3.33) = (-3.33)^3 + 10(-3.33)^2 + 31(-3.33) + 30$

$= 0.73$



8. $f(x) = x^3 - 3x^2 - x + 3$

$f'(x) = 3x^2 - 6x - 1$

$0 = 3x^2 - 6x - 1$

$$x^* = \frac{+6 \pm \sqrt{(-6)^2 - 4(3)(-1)}}{2(3)}$$

$= +2.15, -0.15$

$f(2.15) = (2.15)^3 - 3(2.15)^2 - (2.15) = 3$

$= -3.08$

$f(-0.15) = (-0.15)^3 - 3(-0.15)^2 - (-0.15) + 3$

$= +3.09$

$f''(x) = 6x - 6$

$f''(2.15) = 6(2.15) - 6$

$= +6.9$ (min)

$f''(-0.15) = 6(-0.15) - 6$

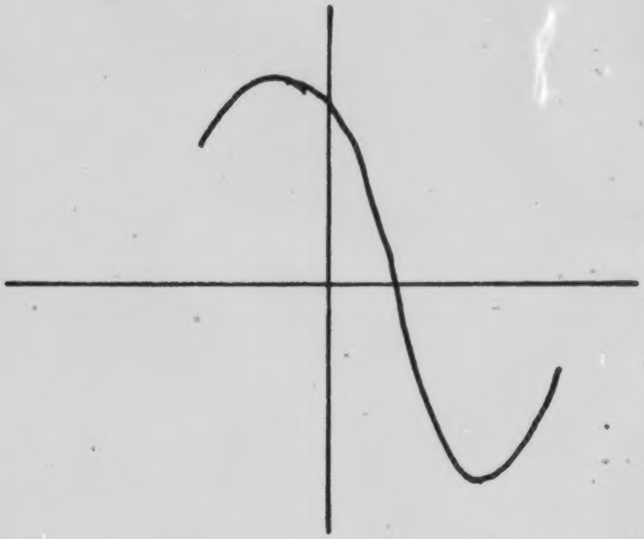
$= -6.09$ (max)

$0 = 6x - 6$

$x^{**} = 1$

$f(1) = (1)^3 - 3(1)^2 - (1) + 3$

$= 0$



9. $f(x) = x^3 - 18^2 + 99x - 162$

$f'(x) = 3x^2 - 36x + 99$

$0 = 3x^2 - 36x + 99$

$0 = 3(x^2 = 12x + 33)$

$$x^* = \frac{12 \pm \sqrt{(-12)^2 - 4(-12)(33)}}{2}$$

$= +26.78, -7.39$

$f(26.78) = (26.78)^3 - 18(26.78)^2 + 99(26.78) - 162$

$= 8785.96$

$f(-7.39) = (-7.39)^3 - 18(-7.39)^2 + 99(-7.39) - 162$

$= -2280.21$

$f''(x) = 6x - 36$

$f''(26.78) = 6(26.78) - 36$

$= + 124.68$ (min)

$f''(-7.39) = 6(-7.39) - 36$

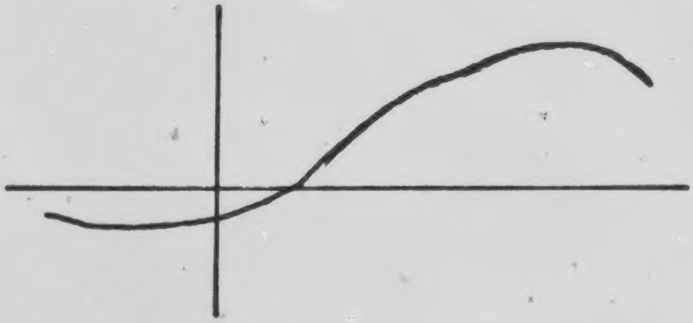
$= -80.34$ (max)

$0 = 6x - 36$

$x^{**} = 6$

$f(6) = (6)^3 - 18(6)^2 + 99(6) - 162$

$= 0$



10. $f(x) = 7x^3 - 22x^2 + 7x - 22$

$f'(x) = 21x^2 - 44x + 7$

$0 = 21x^2 - 44x + 7$

$$x^* = \frac{44 \pm \sqrt{44^2 - 4(21)(7)}}{2(21)}$$

$= +0.173,\ 1.922$

$f(0.173) = 7(0.173)^3 - 22(0.173)^2 + 7(0.173) - 22$

$= -21.41$

$f(1.922) = 7(1.922)^3 - 22(1.922)^2 + 7(1.922) - 22$

$= -40.12$

$f''(x) = 42x - 44$

$f''(0.173) = 42(0.173) - 44$

$= -36.734$ (max)

$f''(1.922) = 42(1.922) - 44$

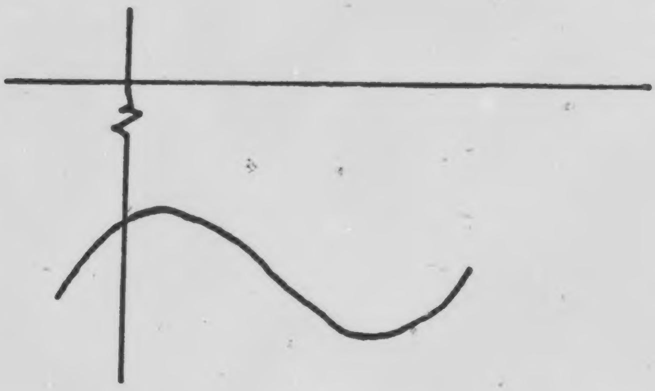
$= 36.724$ (min)

$42x - 44 = 0$

$x^{**} = 1.048$

$f(1.048) = 7(1.048)^3 - 22(1.048)^2 + 7(1.048) - 22$

$= -30.77$



11. $f(x) = x^3 + 6x^2 + 5x - 6$

$f'(x) = 3x^2 + 12x + 5$

$0 = 3x^2 + 12x + 5$

$$x^* = \frac{-12 \pm \sqrt{12^2 - 4(3)(5)}}{2(3)}$$

$= -3.53, -0.47$

$f(-3.53) = (-3.53)^3 + 6(-3.53)^2 + 5(-3.53) - 6$

$= 7.13$

$f(-0.47) = (-0.47)^3 + 6(-0.47)^2 + 5(-0.47) - 6$

$= -7.13$

$f''(x) = 6x + 12$

$f''(-3.53) = 6(-3.53) + 12$

$= -9.18$ (max)

$f''(-0.47) = 6(-0.47) + 12$

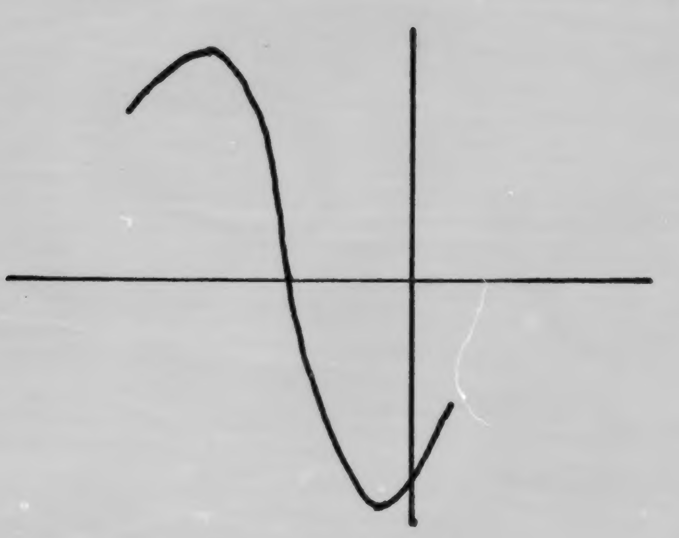
$= +9.18$ (min)

$0 = 6x + 12$

$x^{**} = -2$

$f(-2) = (-2) + 6(-2) + 5(-2) - 6$

$= 0$



12. $f(x) = x^4 - 12x^3 + 52x^2 - 96x + 80$

$f'(x) = 4x^3 - 36x^2 + 10x - 96$

$0 = 4(x^3 - 9x^2 + 26x - 24)$

let $s = +2$ $f'(2) = 4(8 - 36 + 52 - 24) = 0$,

thus, $(x - 2)$ is a factor

$$\begin{array}{r|rrrr}
+2 & 1 & -9 & 26 & -24 \\
 & & 2 & -14 & +24 \\
\hline
+3 & 1 & -7 & +12 & 0 \\
 & & +3 & -12 & \\
\hline
 & 1 & -4 & 0 &
\end{array}$$

Thus $(x - 2)$, $(x - 3)$, and $(x - 4)$ are factors of $f'(x)$

$0 = 4(x - 2)(x - 3)(x - 4)$

$x^* = 2, 3, 4$

$f(2) = 16 - 96 + 208 - 192 + 80 = +16$

$f(3) = 81 - 324 + 468 - 288 + 80 = +17$

$f(4) = 256 - 768 + 832 - 384 + 80 = +16$

$f''(x) = 12x^2 - 72x + 104$

$f''(2) = 48 - 144 + 104 = +8$ (point is a min)

$f''(3) = 108 - 216 + 104 = -4$ (point is a max)

$f''(4) = 192 - 288 + 104 = +8$ (point is a min)

$0 = 12x^2 - 72x + 104$

$$x^{**} = \frac{+72 \pm \sqrt{(-72)^2 - 4(12)(104)}}{2(12)}$$

$= +3.58, +2.42$

$f(3.58) = (3.58)^4 - 12(3.58)^3 + 52(3.58)^2 - 96(3.58) + 80 = 16.2$

$f(2.42) = (2.42)^4 - 12(2.42)^3 + 52(2.42)^2 - 96(2.42) + 80 = 16.4$

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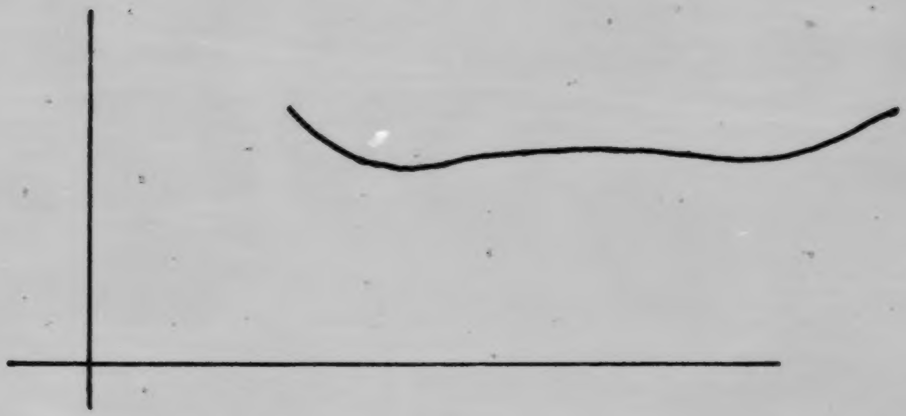


13. $f(x) = x^4 - 20x^3 + 100x^2 + 80$

$f'(x) = 4x^3 - 60x^2 + 200x$

$0 = 4x(x^2 - 15x + 50)$

$= 4x(x - 5)(x - 10)$

$x^* = 0, 5, 10$

$f(0) = 80$

$f(5) = 625 - 2500 + 2500 + 80 = 705$

$f(10) = 10{,}000 - 20{,}000 + 10{,}000 + 80 = 80$

$f'(x) = 12x^2 - 120x + 200$

$f'(0) = +200$ (point is a min)

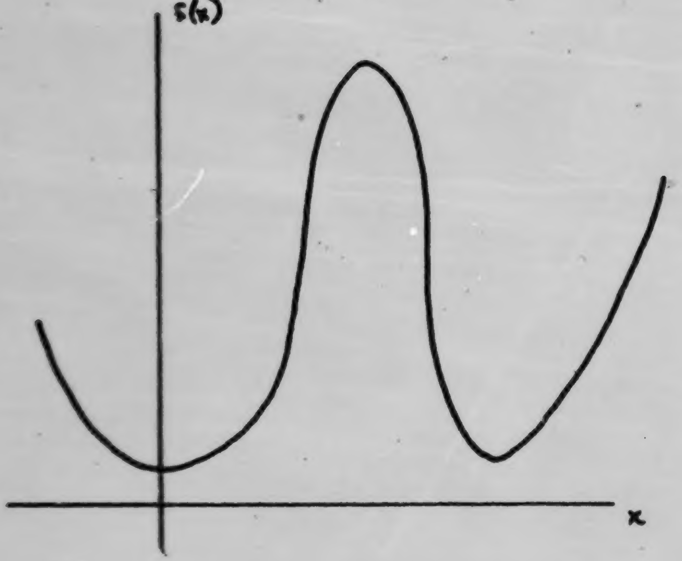
$f''(5) = -1000$ (point is a max)

$f''(10) = +200$ (point is a min)

$0 = 12x^2 - 120x + 200$

$$x^{**} = \frac{+120 \pm \sqrt{120^2 - 4(12)(200)}}{2(12)} = +7.89, +2.11$$

$f(7.89) = 357.8$ $f(2.11) = 357.8$



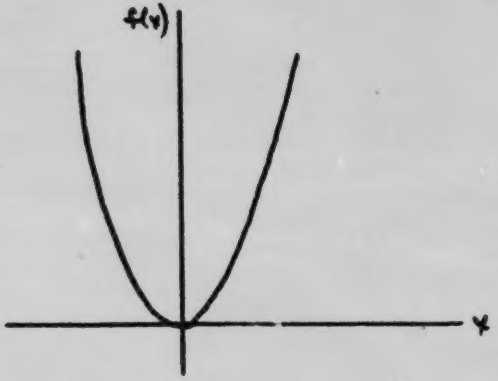
14. $f(x) = x^2$

$f'(x) = 2x$

$0 = 2x$

$x^* = 0$

$f'(x) = +2$



15. $f(x) = x^3$

$f'(x) = 3x^2$

$0 = 3x^2$

$x^* = 0, 0$

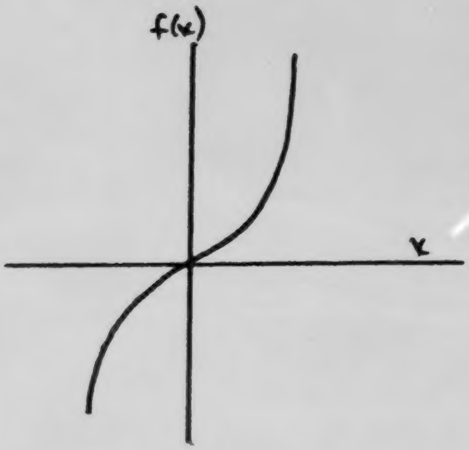
$f(0) = 0$

$f''(x) = 6x$

$f''(0) = 0$

$0 = 6x$

$x^{**} = 0$



16. $f(x) = \ln|x^2 + x|$

$f'(x) = (2x + 1)/(x^2 + x)$

$0 = 2x + 1$

$x^* = -1/2$

$f(-1/2) = \ln|1/4 - 1/2| = \ln|\ 1/4| = -1.38629$

$$f''(x) = \frac{(x^2 + x)(2) - (2x + 1)^2}{(x^2 + x)^2}$$

$$= \frac{-2x^2 - 2x - 1}{(x^2 + x)^2}$$

$$f''(-1/2) = \frac{-2(-1/2)^2 - 2(-1/2) - 1}{[(-1/2)^2 + (-1/2)]^2}$$

$$= \frac{-1/2}{1/16}$$

$$= -8$$

check:

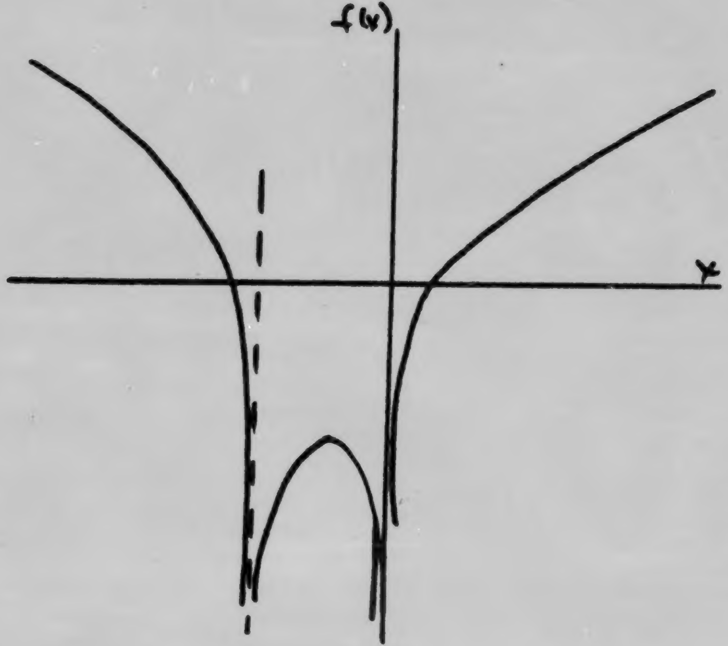
x	-3	-2	-1	-1/2	0	1	2
f(x)	1.79	0.69	und.	-1.39	und.	0.69	1.79

$0 = -2x^2 - 2x - 1$

$0 = -1(2x^2 + 2x + 1)$

$$x^{**} = \frac{+2 \pm \sqrt{2^2 - 4(2)(1)}}{2(2)}$$

$$= \frac{-1 \pm i}{2}$$



17. $f(x) = 3x^4 - 4x^3$

$f'(x) = 12x^3 - 12x^2$

$0 = 12x^2(x - 1)$

$x^* = 0,\ 0,\ 1$

$f(0) = 0$

$f(1) = 3 - 4 = -1$

$f''(x) = 36x^2 - 24x$

$f'(0) = 0$ (horizontal inflection point)

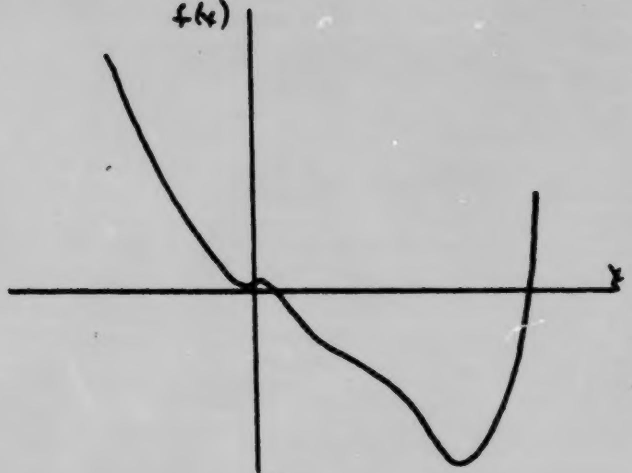
$f''(-1) = +50$ (min)

$0 = 12x(3x - 2)$

$x^{**} = 0,\ 2/3$

$f(0) = 0$

$f(2/3) = 3(2/3)^4 - 4(2/3)^3 = -0.59$



18. $f(x) = x^3/3 - 2x^2 + 3x + 2 \qquad (0 \le x \le 5)$

$f'(x) = x^2 - 4x + 3$

$0 = (x - 1)(x - 3)$

$x^* = 1, 3$

$f(1) = 1/3 - 2 + 3 + 2 = 3.33$

$f(3) = 9 - 18 + 9 + 2 = 2$

$f''(x) = 2x - 4$

$f''(1) = 2(1) - 4 = -2$ (max)

$f''(3) = 2(3) - 4 = +2$ (min)

$0 = 2x - 4$

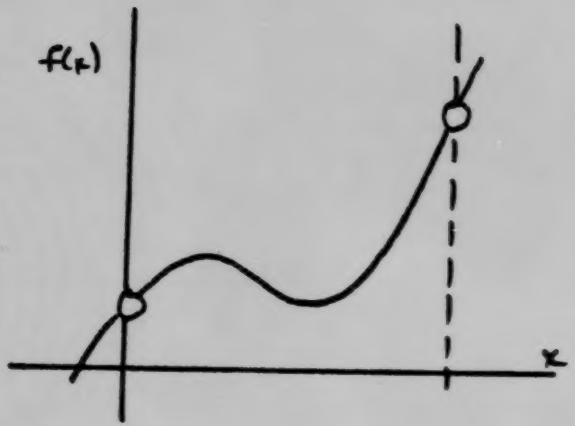
$x^{**} = 2$

$f(2) = 8/3 - 8 + 6 + 2 = 8/3 = 2.67$

global

$f(0) = 2$

$f(5) = 8.67$



19. $f(x) = x^4/4 - 6x^2 + 16x \qquad (0 \le x \le 4)$

$f'(x) = x^3 - 12x + 16$

$0 = x^3 - 12x + 16$

let $s = +2 \quad (2)^3 - 12(2) + 16 = 0$

Thus $(x - 2)$ is a factor. Using synthetic division,

$$\begin{array}{r|rrrr} \hline 2 & 1 & +0 & -12 & +16 \\ & & 2 & 4 & -16 \\ \hline -4 & 1 & 2 & -8 & 0 \\ & & -4 & +8 & \\ \hline & 1 & -2 & 0 & \end{array}$$

$0 = (x - 2)(x + 4)(x - 2)$

$x^* = 2, 2, -4$

$f(2) = (2)^4/4 - 6(2)^2 + 16(2) = 12$

$f(-4) = (-4)^4/4 - 6(-4)^2 + 16(-4) = -96$

$f''(x) = 3x^2 - 12$

$f''(2) = 3(2)^2 - 12 = 0 \qquad$ (horizontal inflection point)

$f''(4) = 3(4)^2 - 12 = +36 \qquad$ (min)

$0 = 3x^2 - 12$

$x^{**} = +2, -2$

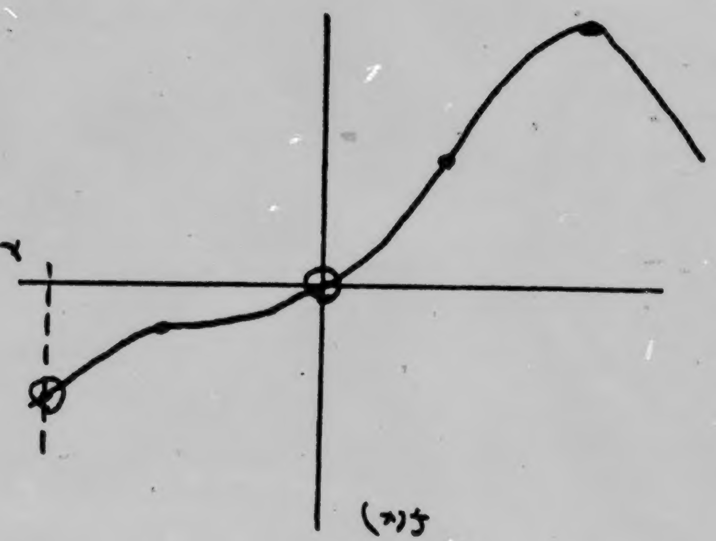
$f(+2) = 12$

$f(-2) = (-2)^2/4 - 6(-2) + 16(-2) = -52$

global

$f(0) = 0$

$f(4) = (4)^4/4 - 6(4)^2 + 16(4) = 32$



20. $f(x) = x^4$ $(-1 \le x \le +1)$

$f'(x) = 4x^3$

$0 = 4x^3$

$x^* = 0, 0, 0$

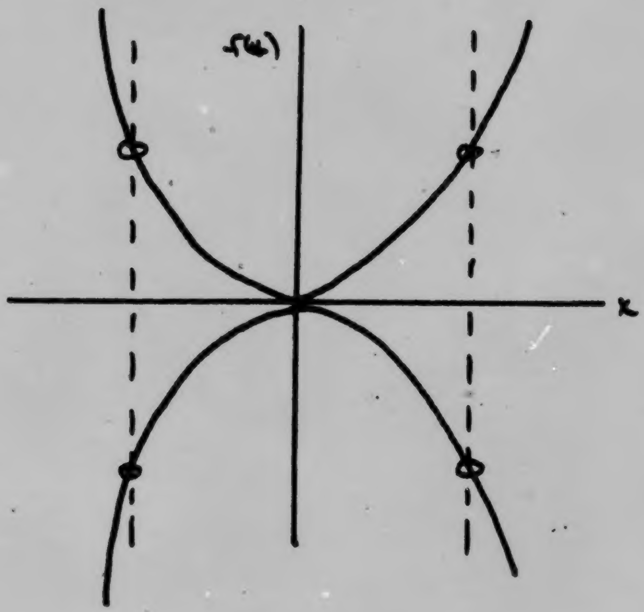
$f(0) = 0$

$f''(x) = 12x^2$

$f''(0) = 0$ (horizontal inflection point)

$0 = 12x^2$

$x^{**} = 0, 0$



21. $f(x) = 2x^3 - 3x^2 + 2$ $(0 \le x \le 2)$

$f'(x) = 6x^2 - 6x$

$0 = 6x(x - 1)$

$x^* = 0, 1$

$f(0) = 2$

$f(1) = 1$

$f''(x) = 12x - 6$

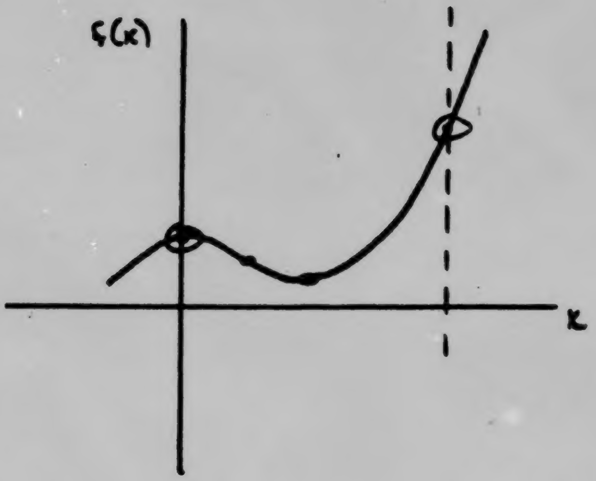
$f''(0) = -6$ (max)

$f''(1) = +6$ (min)

$0 = 12x - 6$

$x^{**} = 1/2$

$f(1/2) = 2(1/2)^3 - 3(1/2)^2 + 2 = 1/4 - 3/4 + 2 = 3/2$



global

$f(0) = 2$

$f(2) = 6$

22. $f(x) = -0.01x^2 + 3x + 200$ $(0 \le x \le 100)$

$f'(x) = -0.02x + 3$

$0 = -0.02x + 3$

$x^* = 150$

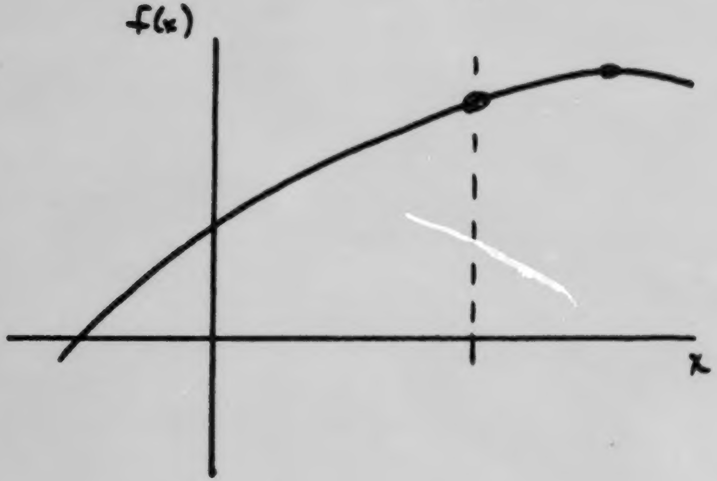
$f(150) = -0.01(150)^2 + 3(150) + 200 = 425$

$f''(x) = -0.02$ (max(

global

$f(0) = 200$

$f(100) = 400$



23. $-2x^3 + 12x^2 - 10x + 5 \qquad (0 \le x \le 5)$

$$f'(x) = -6x^2 + 24x - 10$$

$$0 = -2(3x^2 - 12x + 5)$$

$$x^* = \frac{12 \pm \sqrt{(-12)^2 - 4(3)(5)}}{2(3)}$$

$$= +3.53,\ 0.47$$

$$f(3.53) = -2(3.53)^3 + 12(3.53)^2 - 10(3.53) + 5 = 31.26$$

$$f(0.47) = -2(0.47)^3 + 12(0.47)^2 - 10(0.47) + 5 = 2.74$$

$$f''(x) = -12x + 24$$

$$f''(3.53) = -12(3.53) + 24 = -18.36 \quad \text{(max)}$$

$$f''(0.47) = -12(0.47) + 24 = +18.36 \quad \text{(min)}$$

$$0 = -12x + 24$$

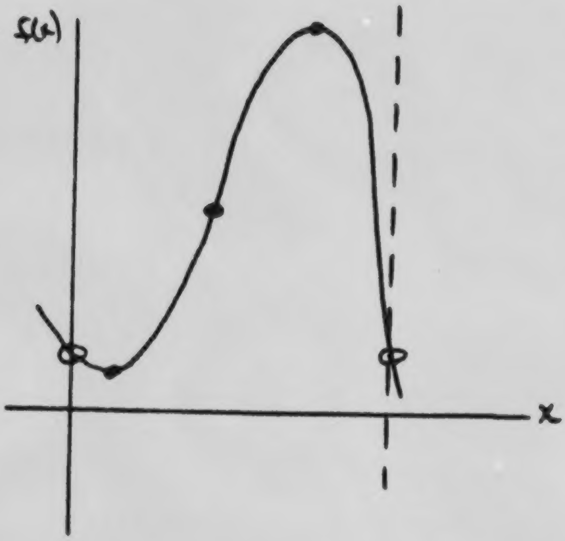
$$x^{**} = 2$$

$$f(2) = -16 + 48 - 20 + 5 = 17$$

global

$$f(0) = 5$$

$$f(5) = -2(125) + 12(25) - 10(5) + 5 = 5$$



24. $f(x) = -2x^3 + 12x^2 - 10$ $(0 \le x \le 6)$

$f'(x) = -6x^2 + 24x$

$0 = -6x(x - 4)$

$x^* = 0,\ 4$

$f(0) = -10$

$f(4) = -2(4)^3 + 12(4)^2 - 10 = 54$

$f''(x) = -12x + 24$

$f''(0) = +24$ (min)

$f''(4) = -12(4) + 24 = -24$ (max)

$0 = -2x + 24$

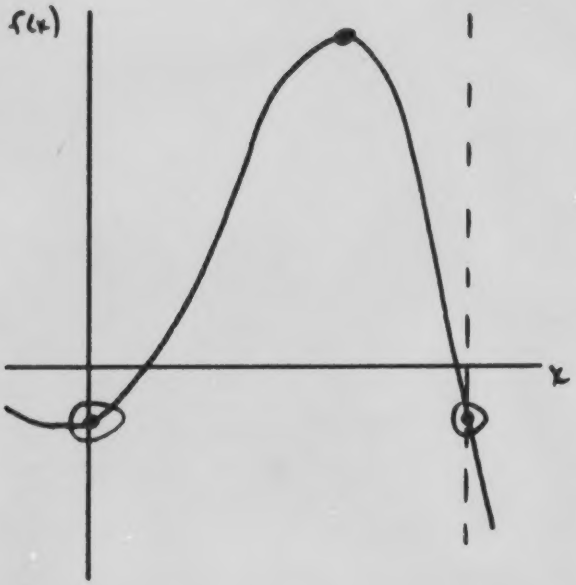
$x^{**} = 2$

$f(2) = -6 + 48 - 10 = 22$

global

$f(0) = -10$

$f(6) = -2(6)^3 + 12(6)^2 - 10 = -10$



25. $f(x) = x^3 - 6x^2 + 9x + 15$ $(0 \le x \le 5)$

$f'(x) = 3x^2 - 12x + 9$

$0 = 3(x^2 - 4x + 3)$

$x^* = 1, 3$

$f(1) = 1 - 6 + 9 + 15 = 19$

$f(3) = 27 - 54 + 27 + 15 = 15$

$f''(x) = 6x - 12$

$f''(1) = 6(1) - 12 = -6$ (max)

$f''(3) = 6(3) - 12 = +6$ (min)

$0 = 6x - 12$

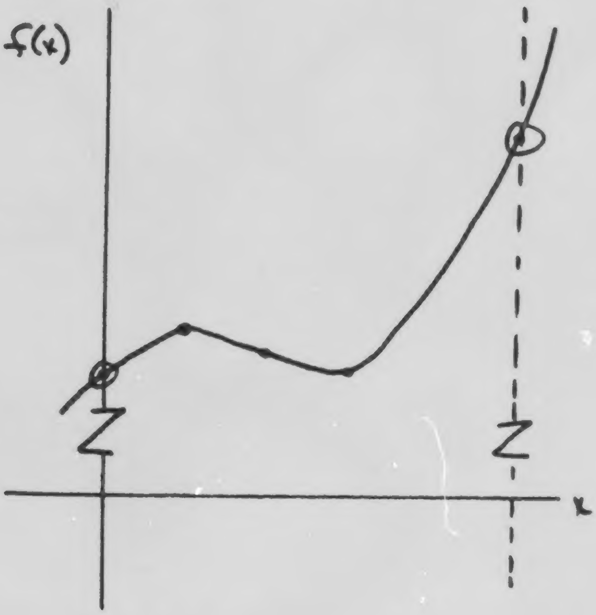
$x^{**} = 2$

$f(2) = 8 - 24 + 18 + 15 = 17$

global

$f(0) = 15$

$f(5) = 125 - 6(25) + 9(5) + 15 = 35$



26. $P(x) = x^3 - 2x^2 - x + 2$

$P'(x) = 3x^2 - 4x - 1$

$0 = 3x^2 - 4x - 1$

$$x = \frac{4 \pm \sqrt{4^2 - 4(3)(-1)}}{2(3)} = +1.55, -0.22$$

$P(1.55) = (1.55)^3 - 2(1.55)^2 - (1.55) + 2 = -0.63$

$P(-0.22) = (-0.22)^3 - 2(-0.22)^2 - (-0.22) + 2 = 2.11$

$P''(x) = 6x - 4$

$P''(1.55) = 6(1.55) - 4 = +5.3$ (min)

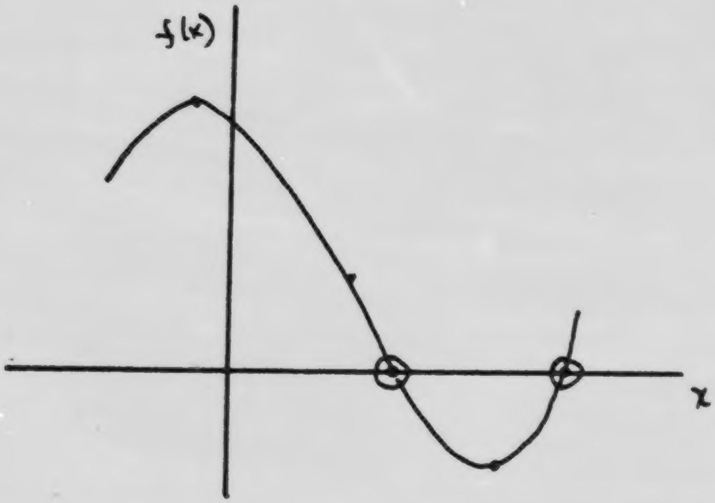
$P''(-0.22) = 6(-0.22) - 4 = -5.32$ (max)

$0 = 6x - 4$

$x^{**} = 2/3 = 0.67$

$f(0.67) = (0.67)^3 - 2(0.67)^2 - (0.67) + 2 = 0.74$

It appears this company had "start-up" money, lost it, and is beginning to show profit again after second unit.



27. $P(x) = -x^3 + 3x + 4$

$P'(x) = -3x^2 + 3$

$0 = -3(x^2 - 1)$

$x^* = -1, +1$

$P(-1) = 1 - 3 + 4 = 2$

$P(+1) = -1 + 3 + 4 = 6$

$P''(x) = -6x$

$P''(-1) = +6$ (min)

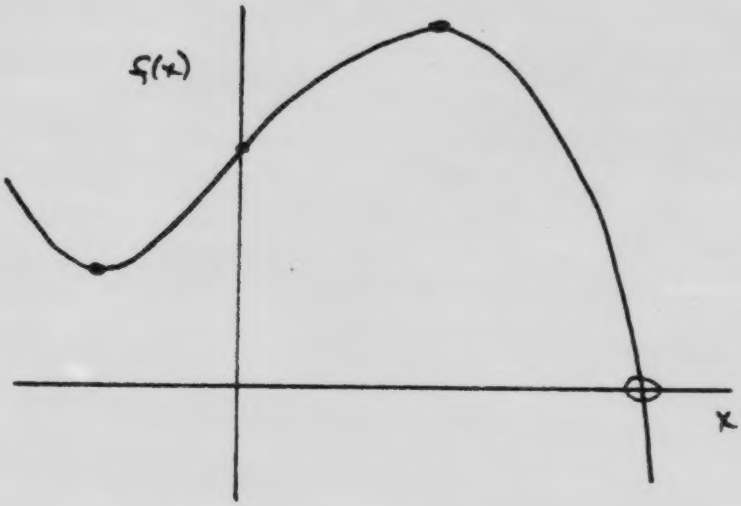
$P''(+1) = -6$ (max)

$0 = -6x$

$x^{**} = 0$

$f(0) = 4$

This company appears to have had "start up" money, realized maximum profit after one month, had a negative profit continuously after 2.2 months.



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28. $D(x) = -x^3 + 18x^2 - 99x + 162$

$D'(x) = -3x^2 + 36x - 99$

$0 = -3(x^2 - 12x + 33)$

$$x^* = \frac{12 \pm \sqrt{144 - 4(1)(33)}}{2} = 7.73,\ 4.23$$

$D(7.73) = -(7.73)^3 + 18(7.73)^2 - 99(7.73) + 162 = 10.39$

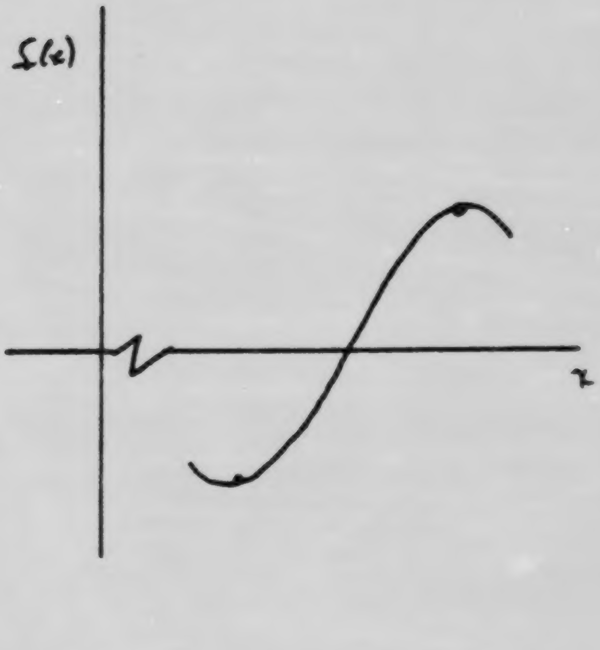
$D(4.23) = -(4.23)^3 + 18(4.23)^2 - 99(4.23) + 162 = -10.38$

$D''(x) = -6x + 36$

$D''(7.73) = -6(7.73) + 36 = -10.38$ (max)

$D''(4.23) = -6(4.23) + 36 = +10.62$ (min)

7.73, or 8 donations, averaging $10.39 each



29. (Demonstration for model building and max-min)

$$f(x) = ax^2 + bx + c$$

$$8000 = a(20)^2 + b(20) + c$$

$$12000 = a(30)^2 + b(30) + c$$

$$14000 = a(40)^2 + b(40) + c$$

$$400a + 20b + c = 8000$$

$$900a + 30b + c = 12000$$

$$1600a + 40b + c = 14000$$

$$|D| = \begin{vmatrix} 400 & 20 & 1 \\ 900 & 30 & 1 \\ 1600 & 40 & 1 \end{vmatrix} = 80,000 - 82,000 = -2000$$

$$a = \frac{\begin{vmatrix} 8000 & 20 & 1 \\ 12000 & 30 & 1 \\ 14000 & 40 & 1 \end{vmatrix}}{-2000} = \frac{1\times10^6 - 0.98\times10^5}{-2000} = -10$$

$$b = \frac{\begin{vmatrix} 400 & 8000 & 1 \\ 900 & 12000 & 1 \\ 1600 & 14000 & 1 \end{vmatrix}}{-2000} = \frac{3.02\times10^7 - 3.20\times10^7}{-2000} = +900$$

$$c = 8000 - (400)(-10) - 20(900) = -6000$$

$$f(x) = -10x^2 + 900x - 6000$$

$$f'(x) = -20x + 900$$

$$0 = -20x + 900$$

$$x^* = 45$$

$$f(45) = -10(45)^2 + 900(45) - 6000 = \$14,250$$

$\frac{d^2y}{dx^2} = -20$ thus x^* is a max

At age 45, the employee has a maximum salary of $14,250.

30. $f(x) = ax^2 + bx + c$

$(1955) \quad 50 = a(1)^2 + b(1) + c$

$(1965) \quad 75 = a(2)^2 + b(2) + c$

$(1975) \quad 110 = a(3)^2 + b(3) + c$

$a + b + c = 50$

$4a + 2b + c = 75$

$9a + 3b + c = 110$

$$|D| = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix} = 23 - 25 = -2$$

$$a = \frac{\begin{vmatrix} 50 & 1 & 1 \\ 75 & 2 & 1 \\ 110 & 3 & 1 \end{vmatrix}}{-2} = \frac{435 - 445}{-2} = 5$$

$$b = \frac{\begin{vmatrix} 1 & 50 & 1 \\ 4 & 75 & 1 \\ 9 & 110 & 1 \end{vmatrix}}{-2} = \frac{965 - 985}{-2} = 10$$

$c = 50 - 5 - 10 = 35$

$f(x) = 5x^2 + 10x + 35$

(a) 1985 $f(4) = 5(4)^2 + 10(4) + 35 = 155$

In 1985, the cost will be \$155

(b) $f(x) = 5x^2 + 10x + 35$

$f'(x) = 10x + 10$

$0 = 10x + 10$

$x^* = -1$

$f''(x) = +10$

Thus, at $x^* = -1$, cost is a relative min.

Actual (global) minimum would have been in 1955 (year records started, x = +1) at \$50. The global max depends on domain of x.

31.

x	A(10)
0	1.4000
1	1.5075
2	1.4600
3	1.3025
4	1.0800
5	0.8375
6	0.6200
7	0.4735
8	0.4400
9	0.5675
10	0.9000
11	1.4825
12	2.3600

$A(x) = 7500x^3 - 100{,}000x^2 + 200{,}000x + 1{,}400{,}000$

$A''(x) = 22{,}500x^2 - 200{,}000x + 200{,}000$

$0 = A'(x)$

$$x^* = \frac{200{,}000 \pm \sqrt{(2\times10^5)^2 - 4(2.25\times10^4)(2\times10^5)}}{2(2.25\times10^4)}$$

$= 1.1484,\ 7.7405$

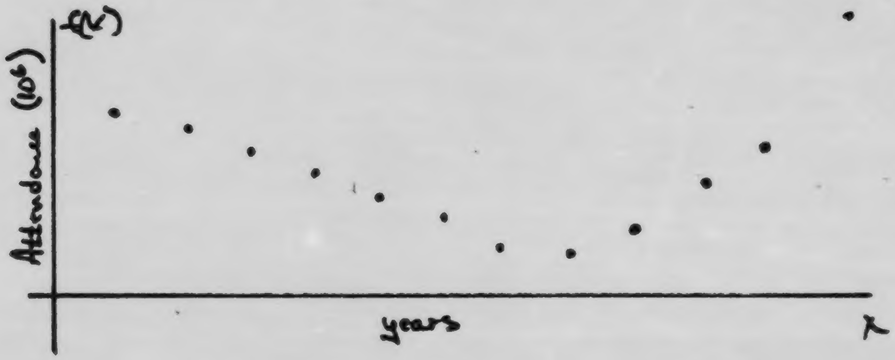
$A''(x) = 45{,}000x - 200{,}000$

$A''(1.1484) < 0$ (max)

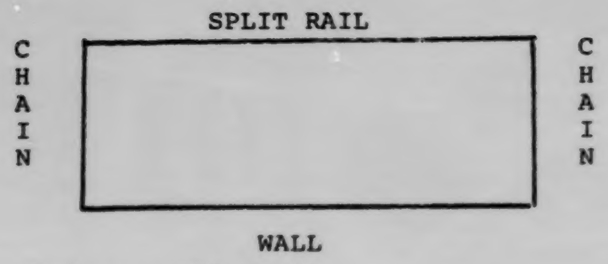
$A''(7.405) > 0$ (min)

Rel min attendance = $A(7.7405) = 0.4349\times10^6$

Rel max attendance = $A(1.1484) = 1.5092\times10^6$



32.



Let y = split rail

x = chain

minimum cost: $2x(\$7.75) + 9/3(y)$

subject to: $xy = 200 \text{ ft}^2$

solving for y: $y = 200/x$

substituting: $\text{min cost} = 15.50x^1 + 3(200/x)^{-1}$

$= 15.50x^1 + 600x^{-1}$

$$\frac{dc}{dx} = 15.50 - 600x^{-2} = 0$$

$x = 600/15.50 = 38.71 \text{ ft}^2$

$x^* = 6.22$ ft

$y = 200/6.22 = 32.15$ ft

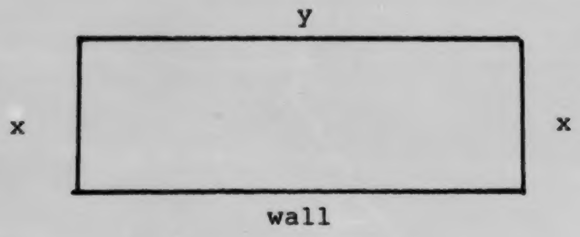
$\text{cost} = 15.50(6.22) + 3.00(32.15)$

$= 96.41 + 96.45$

$= \$192.96$

$$\frac{d^2c}{dx^2} = \frac{1200}{x^3} > 0 \text{ always} \qquad \text{point } \underline{\text{must}} \text{ be a } \underline{\text{minimum}}$$

33.



maximize area = xy

subject to: $2x + y = 50$

solving for y: $y = 50 - 2x$

substituting: Max $A = y(50 - 2x) = 50x - 2x^2$

$$\frac{dA}{dx} = 50 - 4x = 0$$

$$4x = 50$$

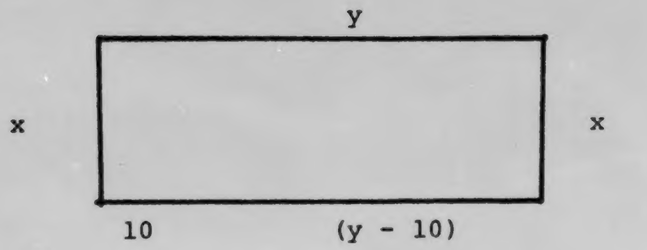
$$x^* = 12.5$$

$$y = 50 - 2(12.5) = 25$$

$A = 25(12.5) = 312.50$ ft^2

$\frac{d^2A}{dx^2} = -4 < 0$ must be a maximum

34.



maximize area = xy

subject to $2x + y + (y - 10) = 50$

or $2x + 2y = 60$

or $x + y = 30$

solving for y: $y = 30 - x$

substituting: Max $A = x(30 - x) = 30x - x^2$

$$\frac{dA}{dx} = 30 - 2x = 0$$

$$x = 15'$$

$$y = 30 - 15 = 15'$$

area = (15)(15) = 225 square feet

$\frac{d^2A}{dx^2} = -2 < 0$ point must be a maximum

Exercise 10.3, pp. 432-433.

1. $P(x) = 100x - 5x^2$

$P'(x) = 100 - 10x$

$x^* = 10$

$P''(x) = -10$ (thus, $x^* = 10$ is a max)

$P(10) = 100(10) - 5(10)^2 = \$500$

$P'(9) = 100 - 10(9) = 10$

$P'(10) = 100 - 10(10) = 0$

$P'(11) = 100 - 10(11) = -10$

2. $C(x) = 0.003x^2 - 0.27x + 7.2$

$C'(x) = 0.006x - 0.27$

$x^* = 45$

$C''(x) = +0.006$ (thus, $x^* = 45$ is the number of units which minimizes cost)

$C'(20) = 0.006(20) - 0.27 = -0.15$

3. $f(t) = 500e^{rt}$

$f(5) = 500e^{.06(5)}$

$= \$674.93$

$f'(t) = 500(0.06)e^{.06t}$

$= 30e^{.06t}$

$f'(5) = 30e^{.06(5)}$

$= \$40.50$

4. $R(x) = 10x - 5x^2/2$ $\qquad$ $C(x) = 10 - 6x + x^2/2 + x^3/3$

$R'(x) = 10 - 5x$ $\qquad$ $C'(x) = -6 + x + x^3$

$$R'(x) = C'(x)$$

$$10 - 5x = -6 + x + x^2$$

$$x^2 + 6x - 16 = 0$$

$$(x + 8)(x - 2) = 0$$

$$x = -8, + 2$$

When $x = +2$, profit is maximized

5. $p(x) = 150 - x/2$

$R(x) = [p(x)]x = (150 - x/2)x = 150x - x^2/2$

$R'(x) = 150 - x$

$C'(x) = 4x$

$$R'(x) = C'(x)$$

$$150 - x = 4x$$

$$x = 30 \qquad \text{(maximizes profit)}$$

$p(30) = 150 - 30/2 = \$135$ (selling price which will maximize profit)

6. $S(t) = 10{,}000e^{-x}$

$S'(t) = -10{,}000e^{-x}$

$S'(4) = -10{,}000e^{(-4)}$

$= -183.16$

7. This problem requires students to consider rate of change of marginal values--a "thinker."

x	P
0	200.00
1	280.63
2	332.88
3	363.35
4	377.92
5	381.75
6	379.28
7	374.23
8	369.60
9	367.67
10	370.00
11	377.43
12	390.08
13	407.35
14	427.92
15	449.75
16	470.08
17	485.43
18	491.60
19	483.67
20	456.00

$P(x) = -0.03x^4 + 1.28x^3 - 17.82x^2 + 97.20x + 200$

$P'(x) = -0.12x^3 + 3.84x^2 - 35.64x + 97.20$

$(x - 5)(x - 9)(x - 18) = 0$

$x^* = 5, 9, 18$

$P''(x) = -0.36x^2 + 7.68x - 35.64$

$$x^{**} = \frac{-7.68 \pm \sqrt{(7.68)^2 - 4(-0.36)(-35.64)}}{2(-0.36)}$$

$= 14.51, 6.82$

$P'''(x) = -0.72x + 7.68$

$P'''(14.51) = -2.77$ max increase

$P'''(6.82) = +2.77$ min increase

8. $S(x) = 90 - 100e^{-0.5x}$

$S'(x) = 50e^{-0.5x}$

$S'(1) = 50e^{-0.5(1)}$

$= 30.33\times10^3$

$S'(2) = 50e^{-0.5(2)}$

$= 18.40\times10^3$

$S'(3) = 50e^{-0.5(3)}$

$= 11.16\times10^3$

9. $f(t) = 1000(1.05)^t$

$f'(t) = 1000(1.05)^t \ln 1.05$

$f'(3) = 1000(1.05)^3 \ln 5 = \$56.48$

Exercise 10.4, p. 437.

$$N_d = -\frac{dq}{dp} \times \frac{p}{q}$$

1. $q = 16 - 4p$

$$\frac{dq}{dp} = -4$$

$$N_d = -(-4)\frac{p}{16 - 4p}$$

$$= \frac{4(1)}{16 - 4} = 1/3 \quad \text{(relatively inelastic)}$$

2. $N_d = \frac{-(-4)p}{16 - 4p} = \frac{+4(2)}{8} = 1$ (unit elasticity)

3. $N_d = \frac{-(-4)p}{16 - 4p} = \frac{+4(3)}{4} = 3$ (relatively elastic)

4. $q = 10 - 100 - p$ (p is in \$)

$$\frac{dq}{dp} = -100$$

$$N_d = -(-100)\frac{p}{(10 - 100p)}$$

(at 4¢) $N_d = \frac{+100(.04)}{10 - 100(.04)} = 4/6 = 2/3$ (relatively inelastic)

(at 5¢) $N_d = \frac{+100\ (.05)}{10 - 100(.05)} = 5/5 = 1$ (unit elasticity)

(at 6¢) $N_d = \frac{+100(.06)}{10 - 100(.06)} = 6/4 = 1.5$ (relatively elastic)

The optimal price is 5¢.

5. $q = 20 - 2p$

$$\frac{dq}{dp} = -2$$

$$N_d = -(-2)\left[\frac{p}{20 - 2p}\right] = 1$$

$2p = 20 - 2p$

$4p = 20$

$p = \$5.00$

6. $N_c = \frac{q}{C} \times \frac{dC}{dq}$

$C(q) = 2q + q^3$ $\qquad$ $C(2) = 2(2) + (2)^3 = 12$

$\frac{dC}{dq} = 2 + 3q^2$ $\qquad$ $C(3) = 2(3) + (3)^3 = 33$

for q = 2: $N_c = \frac{2}{C(2)} \; [2 + 3(2)^2] = 28/12 = 2.33$

(relatively elastic)

for q = 3: $N_c = \frac{3}{C(3)} \; [2 + 3(3)^2] = 3/33(29) = 2.64$

(relatively elastic)

or using

$N_c = MC/AC = \frac{2 + 3q^2}{2 + q^2}$

for q = 2 $N_c = \frac{2 + 3(2)^2}{2 + 2^2} = 14/6 = 2.33$

for q = 3 $N_c = \frac{2 + 3(3)^2}{2 + 3^2} = 29/11 = 2.64$

for unit elasticity:

$$N_c = MC/AC = 1$$

$$\frac{2 + 3q^2}{2 + q^2} = 1$$

$$2 + 3q^2 = 2 + q^2$$

$$q = 0$$

7. $C(x) = -3x^2 + 14x + 2$

$C'(x) = -6x + 14$

$N_c = MC/AC = \frac{-6x + 14}{-3x + 14 + 2/x}$

for x = 1 million $N_c = \frac{-6(1) + 14}{-3(1) + 14 + 2(1)} = 8/13 = 0.62$

for x = 2 million $N_c = \frac{-6(2) + 14}{-3(2) + 14 + 2/2} = 2/9 = 0.22$

In both case, relatively inelastic.

Optimal cost:

$$C'(x) = -6x + 14$$

$$0 = -6x + 14$$

$$x^* = 2.33$$

$$C''(x) = -6 \quad \text{relative max}$$

Thus,

$$C(2.33) = -16.2867 + 32.62 + 2 = \$18.333\times10^6$$

Exercise 10.5, pp. 441-442

1. $I = 50{,}000$

   $a = \$100$

   $c = \$10$

$$Q = +\sqrt{\frac{2(50{,}000)(100)}{10}}$$

$$= 1{,}000 \text{ tires}$$

2. $I = 100$

   $C = \$0.05$

   $a = \$0.40$

$$Q = +\sqrt{\frac{2(100)(0.40)}{0.05}}$$

$$= 40 \text{ cards}$$

3. $Q = 10$

   $I = 1000$

   $C = \$0.20$

$$10 = +\sqrt{\frac{2(1000)a}{0.20}}$$

$$100 = \frac{2(1000)a}{0.20}$$

$$a = \$0.01$$

4. $a = \$10$

$I = 500$

$a = CQ^2/2$

$$C = 10(I)/Q + (0.0001/2)Q^2 + 0.10(I)$$

$$C = \frac{10(500)}{Q} + 0.00005Q^2 + 0.10(500)$$

$$\frac{dC}{dQ} = \frac{-5000}{Q^2} + 0.0001Q$$

$$0 = -5000/Q^2 + 0.0001Q$$

$$Q^3 = 5000/0.0001$$

$$Q = 368.40$$

$$C = \frac{10(500)}{368.40} + 0.00005(368.40)^2 + 0.10(500) = \$131.43 \text{ per week}$$

$$\frac{d^2C}{dQ^2} = \frac{10,000}{Q^3} + 0.0001 > 0 \quad \text{Always (minimum)}$$

CHAPTER 11

MULTIVARIABLE DIFFERENTIAL CALCULUS

This very brief chapter introduces the student to the concepts of multivariable differential calculus. Partial differential calculus is presented with a work-setting example.

Constraints are discussed and the use of Lagrange's multiplier is employed. A clear presentation of this complex topic is given in example 4.

Exercise 11.2, pp. 450-451.

1. $C(x,y) = 2x^2 - 50x - 75y + 1200 + 3y^2 + 3xy$

$C_x = 4x - 50 + 3y$

$C_{xx} = 4$

$C_{xy} = 3$

$C_y = -75 + 6y + 3x$

$C_{yy} = 6$

$C_{yx} = 3$

$D(x,y) = 4(6) - 3^2$

$= +15$

$C = +4$ (minimum)

$4x + 3y = 50$

$3x + 6y = 75$

$5x = 25$

$x = 25$

$x = 5$ (workers with spray guns)

$4(5) + 3y = 50$

$3y = 30$

$y = 10$ (workers with paint brushes)

$C(5,10) = 2(5)^2 - 50(5) - 75(10) + 1200 + 3(10)^2 + 3(5)(10)$

$= \$700.00$

2. $P(x,y) = -5x^2 + 4xy - 30x + 100y - 3y^2 + 75$

$P_x = -10x + 4y - 30$

$P_{xx} = -10$

$P_y = 4x + 100 - 6y$

$P_{yy} = -6$

$P_{xy} = 4$

$P_{yx} = 4$

$D_{xy} = -10(-6) - 4^2$

$= +56$

$P_{xx} = -10$ (maximum)

$-10x + 4y = 30$

$4x - 6y = -100$

$-11y = -220$

$y = 20$

$-10x + 4(20) = 30$

$x = 5$

$P(5,20) = 5(5)^2 + 4(5)(20) - 30(5) + 100(20) - 3(20)^2 + 75$

$= \$1000.00$

3. $f(x,y) = x^3 - 3x + y^2$

$f_x = 3x^2 - 3$

$f_{xx} = 6x$

$f_y = 2y$

$f_{yy} = 2$

$f_{xy} = 0$

$f_{yx} = 0$

$3x^2 = 3$

$2y = 0$

$3x^2 + 2y = 3$

$y = 3/2\ x^2 - 3/2$

$x = \pm 1$ points are (1,0), (-1,0)

$D_{xy} = (6x)(2) - 0^2$

$= 6(+1)(2) - 0^2$

$= +12$

$f_{xx} = 6(1) = +6$ minimum [point (1,0) is a minimum]

$6(-1)(2) - 0^2 = -12$ [point (-1,0) is a saddle point]

4. $f(x,y) = -2y + 3xy + x^2 - 20$

$f_x = 3y + 2x$

$f_{xx} = 2$

$f_y = -2 + 3x$

$f_{yy} = 0$

$f_{xy} = 3$

$f_{yx} = 3$

$D_{xy} = 2(0) - 3^2$

$= -9$ (saddle point)

$2x + 3y = 0$

$3x = 2$

$x = 2/3$

$2(2/3) + 3y = 0$

$3y = -4/3$

$y = - 4/9$

5. $P(x,y) = -3x^2 - 2x/y + 20x - 30y$

$P_x = -6x - 2/y + 20$

$P_{xx} = -6$

$P_y = 2x/y^2 - 30$

$P_{yy} = -4x/y^3$

$P_{xy} = 2/y^2$

$P_{yx} = 2/y^2$

$-6x - 2/y = -20$

$-2x/y^2 = 30$

$-6xy - 2 = -20y$

$2x = 30y^2$

$x = 15y^2$

$-6(15y^2) - 2 = -20y$

$-90y^3 + 20y - 2 = 0$

[Requires Newton's method: please refer to Study Guide, pp. 170-174.]

$y = 0.41, 0.105, -0.515$

$x = 15(0.41)^2$

$= 2.52$

Thus $x = \$2520$ advertising budget

$y = \$410$ competitor's budget

$D_{xy} = -6(-4x/y^3) - (2/y^2)^2$

$= -6[-4(2.52)/.41^3] - (2/.41^3)$

$= +735.97$

$f_{xx} = -6$ (maximum)

Exercise 11.3, p. 456.

1. $C(x,y) = 2x^2 - 5xy + 4y^2 + 10x - 30y$

$g(x,y) = x + y - 15$

$C(x,y,\lambda) = 2x^2 - 5xy + 4y^2 + 10x - 30y + \lambda(x + y - 15)$

$C_x = 4x - 5y + 10 + \lambda$

$C_{xx} = 4$

$C_y = -5x + 8y - 30 + \lambda$

$C_{yy} = 8$

$C_\lambda = x + y - 15$

$C_{xy} = -5$

$C_{yx} = -5$

$D_{xy} = 4(8) - (-5)^2$

$= 7$

$C_{xx} = 4$ (minimum)

$4x - 5y + \lambda = -10$

$-5x + 8y + \lambda = 30$

$x + y = 15$

$9x - 13y = -40$

$x + y = 15$

$-22y = =175$

$y = 7.95$

$x + 7.95 = 15$

$x = 7.05$

2. $f(x,y) = 3x^2 + xy + y^2 - 15x - 7y$

$g(x) = x + y$

$F(x,y,\lambda) = 3x^2 + xy + y^2 - 15x - 7y + \lambda(x + y)$

$F_x = 6x + y - 15 + \lambda$

$F_{xx} = 6$

$F_y = x + 2y - 7 + \lambda$

$F_{yy} = 2$

$F_{xy} = 1$

$F_{yx} = 1$

$F_\lambda = x + y$

$D_{xy} = (6)(2) - 1^2$

$= +11$

$F_{xx} = +6$ (minimum)

$6x + y + \lambda = 15$

$x + 2y + \lambda = 7$

$x + y = 0$

$5x - y = 8$

$x + y = 0$

$6x = 8$

$x = 1.33$

Thus, $y = -1.33$

3. $R(x,y) = -3x^2 + 36x + 3xy - 16y - y^2 + 128$

$g(x,y) = x + y - 10$

$R(x,y,\lambda) = -3x^2 + 36x + 3xy - 16y - y^2 + 128 + \lambda(x + y - 10)$

$R_x = -6x + 36 + 3y + \lambda$

$R_{xx} = -6$

$R_y = 3x - 16 - 2y + \lambda$

$R_{yy} = -2$

$R_{xy} = 3$

$R_{yx} = 3$

$R_\lambda = x + y - 10$

$D_{xy} = (-6)(-2) - 3^2$

$= +3$

$R_{xx} = -6$ (maximum)

$-6x + 3y + \lambda = -36$

$3x - 2y + \lambda = 16$

$x + y = 10$

$-9x + 5y = -52$

$x + y = 10$

$14y = 38$

$y = 2.72$

$x + 2.72 = 10$

$x = 7.28$

$R(2.72, 7.28) = -3(2.72)^2 + 36(7.28) + 3(2.72)(7.28) - 16(7.28)$

$- (7.28)^2 + 128$

$= \$257.81$

4. $P(x,y) = -40x^2 + 500x + 1000 - 20y^2 + 200y + 200$

$g(x,y) = x + y - 10$

$P(x,y,\lambda) = -40x^2 + 500x + 1000 - 20y^2 + 200y + 200 + \lambda(x + y - 10)$

$P_x = -80x + 500 + \lambda$

$P_{xx} - 80$

$P_y = - 40y + 200 + \lambda$

$P_{yy} = - 40$

$P_{xy} = 0$

$P_{yx} = 0$

$P_\lambda = x + y - 10$

$D_{xy} = (-80)(-40) - 0^2$

$= + 3200$

$P_{xx} = -80$ (maximum)

$-80x + \lambda = -500$

$- 40y + \lambda = -200$

$x + y = 10$

$-80x + 40y = -300$

$x + y = 10$

$-120x = -700$

$x = 5.83$

$5.83 + y = 10$

$y = 4.17$

$P(5.83, 4.17) = 40(5.83)^2 + 500(5.83) + 1000 - 20(4.17)^2 + 200(4.17) + 200$

$= \$5041.56$

CHAPTER 12

INTEGRAL CALCULUS

Before introducing the basic rules of integral calculus, a discussion of area by summation prepares the student. By using 55 examples, the basic rules, integration by substitution, and definite and improper integrals are presented.

In the definite integral section, techniques studied in Chapter 6 are readily employed (see example 50 on pages 486-488). To calculate the area contained between two functions requires insight on the student's part. This leads directly into setting the stage for applications in Chapter 13. The student is encouraged to sketch the functions, using techniques learned in max-min, to visualize the area being calculated.

Exercise 12.4, pp. 471-472.

1. $3x + c$
2. $20x + c$
3. $-4x + c$
4. $x + c$
5. $1/3x + c$
6. $4/5x + c$
7. $-0.5x + c$
8. $x^3/3 + c$
9. $x^5/5 + c$
10. $x^9/9 + x$
11. $-x^7/7 + c$
12. $-x^{-1} + c = -1/x + c$
13. $\frac{3x}{5}^{5/3} + c$
14. $\frac{7x^{13/7}}{13} + c$
15. $3x^{1/3} + c$
16. $-4x^{-1/4} + c$
17. $1.25x^{.8} + c$
18. $x^{2.3}/2.3 + c$
19. $x^{101}/101 + c$
20. $5x^3/3 + c$
21. $x^4 + c$
22. $7x^5/5 + c$
23. $x^4/16 + c$
24. $-1/2x + c$
25. $-7x^{-2}/16 + c$
26. $9x^{4/3}/4 + c$
27. $-5x^{4/9}/4 + c$
28. $9 \ln|x| + c$
29. $4/7 \ln|x| + c$
30. $\ln|x| + c$
31. $5 \ln|x| + c$
32. $3/4 \ln|x| + c$
33. $x^2 + x^4/4 + c$
34. $2x - x^3/3 + x^4/4 + c$
35. $4x^3/3 - 12x + c$
36. $x^2/2 + x + \ln|x| + c$
37. $x^2 + 4x^2/3 + c$
38. $\frac{1}{2 \ln 5} (5^{2x}) + c$

39. $\left(\frac{3}{\ln 3}\right) 3^{1/3x} + c$

40. $\left(\frac{1}{\ln 1/2}\right)(1/2^{x}) + c$

41. $\left(\frac{1}{2 \ln [4/5]}\right) 4/5^{2x} + c$

42. $(1/5)e^{5x} + c$

43. $2e^{1/2x} + c$

44. $(2/3)e^{3x} + c$

45. $e^{5x}/5 - 5^{x}/\ln 5 + c$

46. $ex^{e-1} + c$

47. $e^{x} + ex^{e-1} + c$

48. $4 \ln|x| + \frac{5^{x}}{\ln 5} - x^{4}/4 + c$

49. $x^{2} - x^{2}/6 + 5^{x}/\ln x + c$

$= 5x^{2}/6 + 5^{x}/\ln + c$

50. $-6x^{-2} - x^{3}/3 + e^{2x}/2 + c$

Exercise 12.5, p. 482.

1. $\int x^{2} \ln x \, dx$ (See example 30 on page 473 and formula 37 on page 633.)

$u = \ln x$ $dv = x^{2}dx$

$du = 1/x \, dx$ $v = x^{3}/3$

$$\int \ln x \, x^{2}dx = \ln x \, (x^{3}/3) - \int (x^{3}/3)(1/x) \, dx$$
$$= \frac{x^{3}}{3} \ln - \frac{1}{9} x^{3} + c$$

2. $\int \frac{2x \, dx}{x^{2} - 5} = \ln|x^{2} - 5| + c$

3. $\int x^2e^xdx = x^2e^x - 2\int xe^xdx$

$= x^2e^x - 2e^x(x - 1) + c$

$= e^x(x^2 - 2x + 2) + x$

(See formulas 34 and 33 on page 633.)

4. $\int x(x^2 + 3)^4dx = 1/2\int (2)x(x^2 + 3)^4dx$

$= 1/2 \left(\frac{(x^2 + 3)^5}{5}\right) + c$

$= \frac{(x^2 + 3)^5}{10} + c$

5. $\int \frac{x^2 + 1}{x^3 + 3x} dx = 1/3 \int \frac{3(x^2 + 1)}{x^3 + 3x} dx$

$= 1/3 \ln |x^3 + 3x| + c$

6. $\int (x + 2)(x^2 + 4x - 5)^3dx$

$= 1/2 \int (2)(x + 2)(x^2 + 4x - 5)^3 dx$

$= \frac{(x^2 + 4x - 5)^3}{6} + c$

7. $\int x^2(x^3 + 3)^2dx = 1/3 \int (3)(x^2)(x^3 + 3)^2 dx$

$= \frac{(x^3 + 3)^3}{9} + c$

8. $\int \frac{2}{x + 3} dx = 2 \ln|x + 3| + c$

$= \ln (x + 3)^2 + c$

9. $\int (x + 1)(x - 3)^2dx$

$= \int (x + 1)(x^2 - 6x + 9) dx$

$= \int (x^3 - 5x^2 + 3x + 9) dx$

$= \frac{x^4}{4} - \frac{5x^2}{3} + \frac{3x^2}{2} + 9x + c$

10. $\int (x + 7)^3 dx = \frac{(x + 7)^4}{4} + c$

11. $\int \frac{x + 1}{(2x^2 + 4x)^{1/2}} \quad dx$

$= \int (x + 1)(2x^2 + 4x)^{-1/2} \; dx$

$= 1/4 \int 4(x + 1)(2x \;\; + 4x)^{-1/2} \; dx$

$= \frac{1/4(2x^2 + 4x)^{1/2}}{1/2} + c \qquad = \frac{(2x^2 + 4x)^{1/2}}{2} + c$

12. $\int x \, e^{2x} \, dx = \frac{e^{2x}}{4} \; (2x - 1) + c$

(See formula 33 on page 633.)

13. $\int (x + 2)^2 (x - 3)^3 \; dx$

$= \int (x^2 + 4x + 4)(x^3 - 9x^2 + 27x - 27) \; dx$

$= \int (x^5 - 5x^4 - 5x^3 + 45x^2 - 108) \; dx$

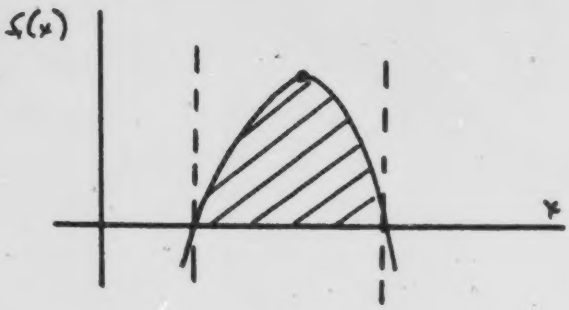
$= \frac{x^6}{6} - x^5 - \frac{5x^4}{4} + 15x^3 - 108x \;\; + x$

14. Same as number 3!

Exercise 12.6, pp. 491-492.

1. $f(x) = -40.5 + 18x - 3/2x^2 \qquad (3 \leq x \leq 9)$

graphing the function:



$f'(x) = 18 - 3x$

$x^* = 6$

$f(6) = -40.5 + 18(6)^2 - 3/2(6)^2$

$= 13.5$

$f''(x) = -3$ (a maximum value)

$\int_3^9 f(x) = \int_3^9 (-40.5 + 18x - 3/2x^2) \; dx$

$= (-40.5x + 9x^2 - x^3/2) \Big|_3^9 = 0 - (-54) = 54$

2. $f(x) = 4 + 9x - 6x^2 + x^3$ $(0 \le x \le 4)$

sketching the graph:

$f(x) = x^3 - 6x^2 + 9x + 4$

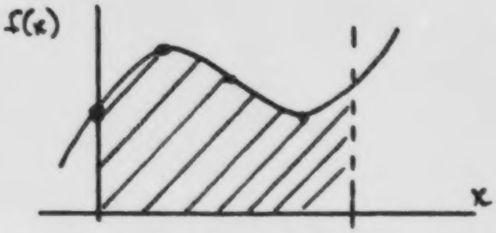
$f'(x) = 3x^2 - 12x + 9$

$0 = 3(x - 1)(x - 3)$

$x^* = 1, 3$

$f(1) = 1 - 6 + 9 + 4 = 8$

$f(3) = 27 - 54 + 27 + 4 = 4$



$f''(x) = 6x - 12$

$f''(1) = -$ (max)

$f''(3) = +$(min)

$x^{**} = 2$ (poi)

$f(2) = 8 - 24 + 18 + 4 = 6$

$$\int_0^4 f(x)\,dx = \int_0^4 (4 + 9x - 6x^2 + x^3)\,dx$$

$$= (4x + 9/2\,x^2 - 2x^3 + x^4/4)\Big|_0^4 = 24$$

3. $f(x) = -1 + x^2$ $(0 \le x \le 1)$

sketching the graph:

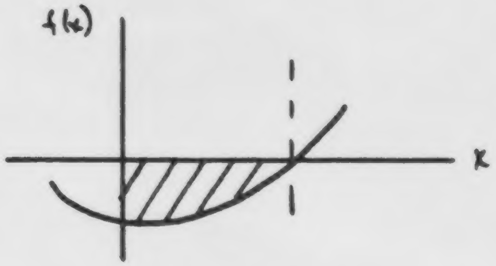
$f(x) = x^2 - 1$

$f'(x) = 2x$

$x^* = 0$

$f(0) = -1$

$f''(x) = +2$ (minimum value)



$$\int_1^0 f(x)\,dx = \int_1^0 (-1 + x^2)\,dx = (-x + x^3/x)\Big|_1^0$$

$$= 0 - (-1 + 1/3) = 2/3$$

4. $f(x) = 10 - x^2$ $(3 \le x \le 9)$

sketching the graph:

$f(x) = -x^2 + 10$

$f'(x) = -2x$

$x^* = 0$

$f(0) = 10$

$f''(x) = -2$ (a maximum value)

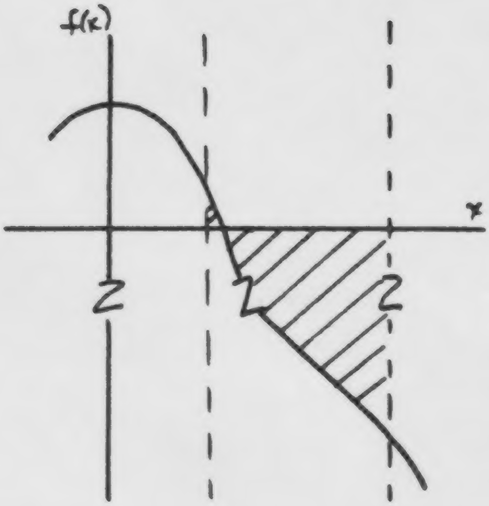
$$\int_3^9 f(x)\,dx = \int_3^{+\sqrt{10}} (10 - x^2)\,dx - \int_{+\sqrt{10}}^{9} (10 - x^2)\,dx$$

$$= (10x - x^3/3)\Big|_3^{\sqrt{10}} - (10x - x^3/3\Big|_{\sqrt{10}}^{9}$$

$$= \left[(10^{3/2} - \frac{10}{3}^{3/2}) - (30 - 9)\right]$$

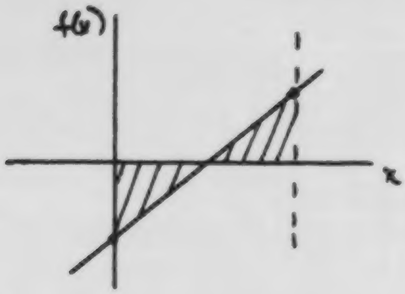
$$- \left[(90 - 243 - (10^{3/2} - \frac{10}{3}^{3/2}))\right]$$

$$= 168.16$$



5. $f(x) = -1/2 + x$ $(0 \le x \le 1)$

sketching the linear equation:



$$\int_0^1 f(x)dx = -\int_0^{1/2} (-1/2 + x)\ dx + \int_{1/2}^1 (-1/2 + x)\ dx$$

$$= -(-1/2x + 1/2\ x^2)\Big|_0^{1/2} + (-1/2x + 1/2x^2)\Big|_{1/2}^1$$

$$= -(-1/4 + 1/8) + [(-1/2 + 1/2) - (-1/4 + 1/8)]$$

$$= -2(-1/8) = 1/4$$

6. $f(x) = x^3 + 10x^2 + 31x + 30$ $(0 \le x \le 3)$

sketching the graph

$f'(x) = 3x^2 + 20x + 31$

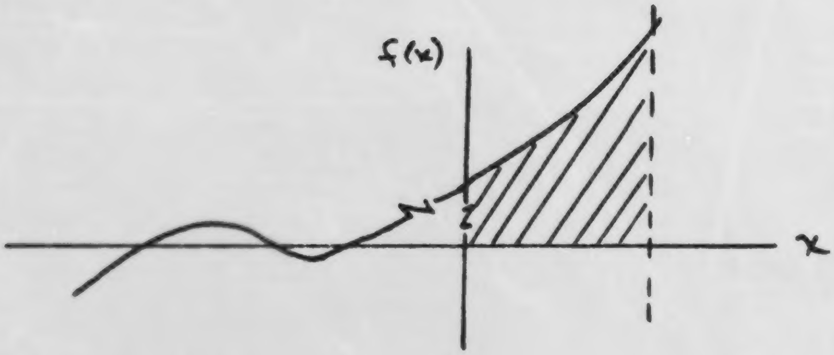
$$x^* = \frac{-20 \pm \sqrt{20^2 - 4(3)(31)}}{2(3)} = -4.22, -2.45$$

$f''(x) = (x + 2)(x + 3)(x + 5)$

$$\int_0^3 f(x)\ dx = \int_0^3 (x^3 + 10x^2 + 3(x + 30)\ dx$$

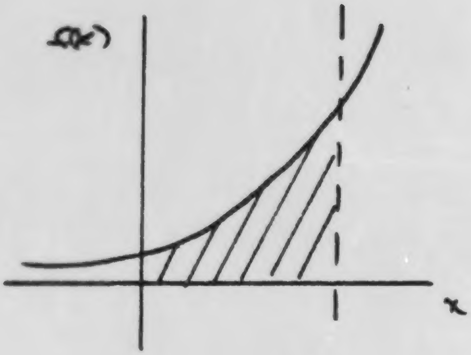
$$= \left(\frac{x^4}{4} + \frac{10x^3}{3} + \frac{31x^2}{2} + 30x\right) \Big|_0^3$$

$$= 339.75$$



7. $f(x) = 3^x$ $(0 \le x \le 3)$

sketching the graph:



$$\int_0^3 f(x)\,dx = \int_0^3 3^x dx = \frac{3^x}{\ln 3}\Big|_0^3 = 24.6$$

8. $f(x) = x^3/3 - 4x^2 + 15x + 5$ $(0 \le x \le 5)$

sketching the graph:

$f'(x) = x^2 - 8x + 15$

$0 = (x - 3)(x - 5)$

$x^* = 3, 5$

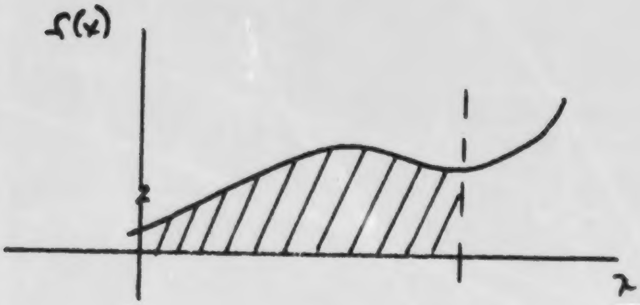
$f(3) = 9 - 36 + 45 + 5 = 23$

$f(5) = 125/3 - 100 + 75 + 5 = +21.667$

$f''(x) = 2x - 8$

$f''(3) = -$ (max)

$f''(5) = +$ (min)



$$\int_0^5 f(x)\,dx = \int_0^5 (x^3/3 - 4x^2 + 15x + 5)\,dx$$

$$= (x^4/12 - 4x^3/3 + 15x^2/2 + 5x)\Big|_0^5$$

$$= 670.83$$

9. $f(x) = x^2 - 8x + 15$ $(0 \le x \le 8)$

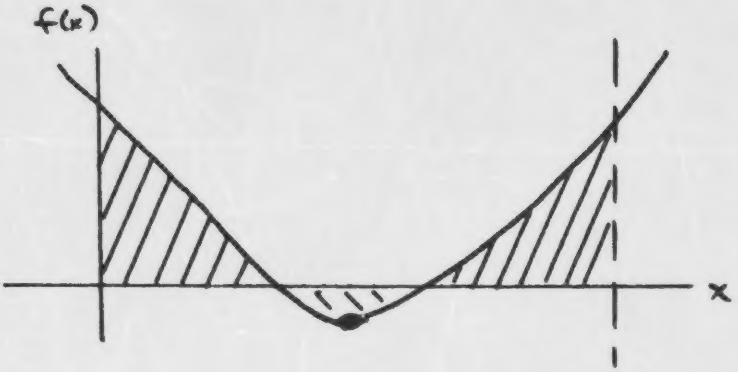
sketching the graph:

$f'(x) = 2x - 8$

$x^* = 4$

$f(4) = 16 - 32 + 15 = -1$

$f''(x) = +2$ (a minimum value)



$$\int_0^8 f(x)\,dx = \int_0^3 (x^2 - 8x + 15)\,dx - \int_3^5 (x^2 - 8x + 15)dx + \int_5^8 (x^2 - 8x + 15)\,dx$$

$$= (x^3/3 - 4x^2 + 15x)\Big|_0^3 - (x^3/3 - 4x^2 + 15x)\Big|_3^5 + (x^2/3 - 4x^2 + 15x)\Big|_5^8$$

$$= (9 - 36 + 45) - [(125/3 - 100 + 75) - (9 - 36 + 45)]$$

$$+ [(512/3 - 256 + 120) - (125/3 - 100 + 75)]$$

$$= 37.33$$

10. $f(x) = x^3 - 18x^2 + 99x - 162 \qquad (3 \le x \le 9)$

sketching the function:

$$f'(x) = 3x^2 - 36x + 99$$

$$0 = 3(x^2 - 12x + 33)$$

$$x^* = \frac{12 \pm \sqrt{(-12)^2 - 4(1)(33)}}{2} = 7.73,\ 4.27$$

$$f(7.73) = -10.39$$

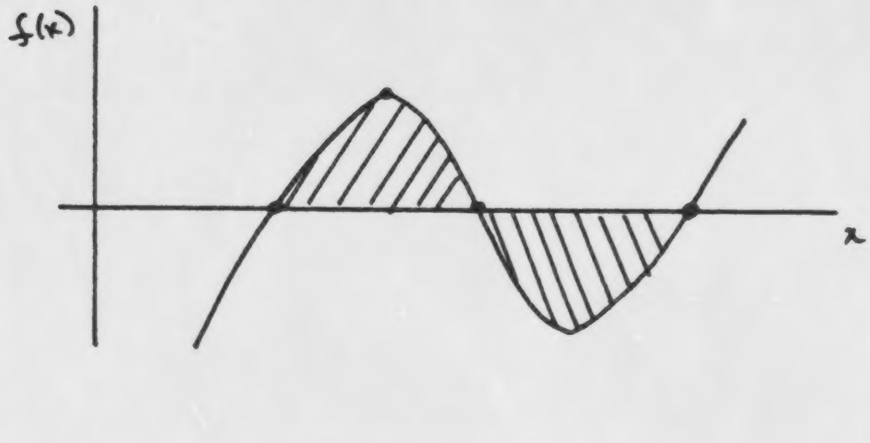
$$f(4.27) = +10.39$$

$$f''(x) = 6x - 36$$

$$f''(7.73) = + \quad \text{(min)}$$

$$f''(4.27) = - \quad \text{(max)}$$

$$f(x) = (x - 3)(x - 6)(x - 9)$$



$$\int_3^9 f(x)\,dx = \int_3^6 (x^3 - 18x^2 + 99x - 162)dx - \int_6^9 (x^3 - 18x^2 + 99x - 162)\,dx$$

$$= (x^4/4 - 6x^3 + 99x^2/2 - 162x)\Big|_3^6 - (x^4/4 - 6x^3 + 99x^2/2 - 162x)\Big|_6^9$$

$$= [(-162) - (-182.25)] - [(-182.25) - (162)]$$

$$= 2(20.25) = 40.5$$

11. $f(x) = -2x^3 + 12x^2 - 10x$ $(0 \le x \le 6)$

sketching the function:

$f'(x) = -6x^2 + 24x - 10$

$$x^* = \frac{-24 \pm \sqrt{24^2 - 4(-6)(-10)}}{2(-6)} = +3.53,\ +0.47$$

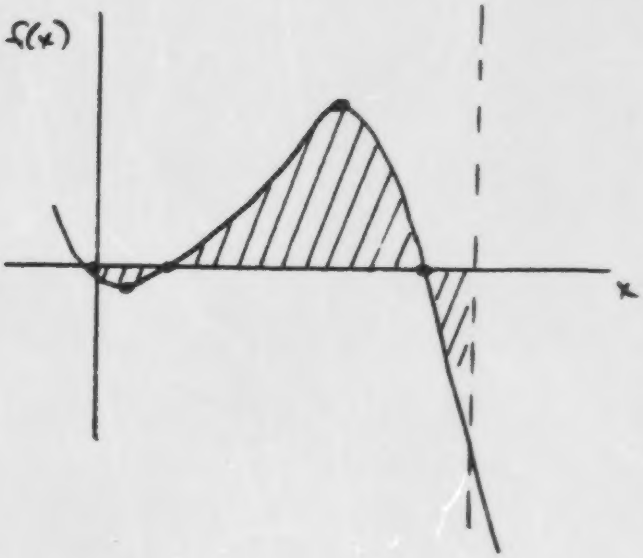
$f(3.53) = 26.3$

$f(0.47) = -2.3$

$f''(x) = -12x + 24$

$f''(3.53) = -$ (max)

$f''(0.47) = +$ (min)



$$\int_0^6 f(x)\,dx = -\int_0^1 (-2x^3 + 12x^2 - 10x)dx + \int_1^5 (-2x^3 + 12x^2 - 10x)dx$$

$$- \int_0^6 (-2x^3 + 12x^2 - 10x)\,dx$$

$$= -[(-x^4/2 + 4x^3 - 5x^2)\Big|_0^1] + [(-x^4/2 + 4x^3 - 5x^2)\Big|_1^5]$$

$$- [(-x^4/2 + 4x^3 - 5x^2)\Big|_5^6]$$

$$= -(-1/2 + 4 - 5) + [(-625/2 + 500 - 125) - (-1/2 + 4 - 5)]$$

$$- [(-648 + 864 - 180) - (-625/2 + 500 - 125)]$$

$$= -3 + 125 + 26.5$$

$$= 148.5$$

12. $f(x) = -x^2$

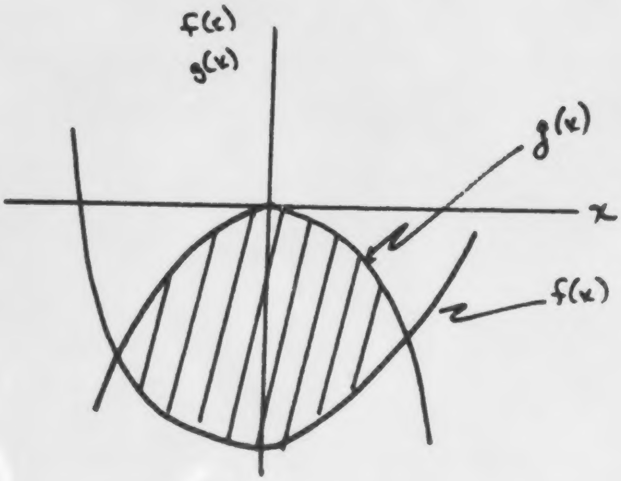
$g(x) = -2 + x^2$

determining limits: $-x^2 = -2 + x^2$

$-2x^2 = -2$

$x = \pm 1$

sketching the function:



$$\text{Area} = -\left(\int_{-1}^{+1} [(-2 + x^2) - (-x^2)]dx\right)$$

$$= -(-2x + 2x^3/3)\Big|_{-1}^{+1}$$

$$= -[(-2 + 2/3) - (2 - 2/3)]$$

$$= (-4 + 4/3)$$

$$= 8/3$$

13. $f(x) = 7 - 1/2x$ $\qquad g(x) = 1 + 1/2x^2$

determining limits:

$$7 - 1/2x = 1 + 1/2x^2$$

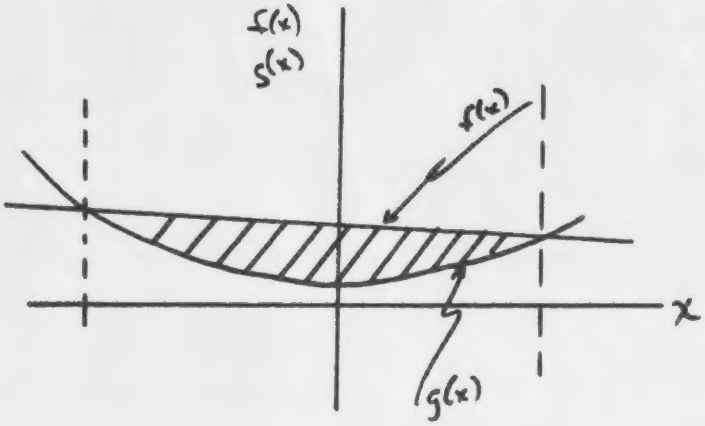
$$1/2x^2 + 1/2x - 6 = 0$$

$$x^2 + x - 12 = 0$$

$$(x + 4)(x - 3) = 0$$

$$x = -4,\ 3$$

sketching the graph:



$$\text{Area} = \int_{-4}^{3} [(7 - 1/2x) - (1 + 1/2x^2)]dx$$

$$= (6x - x^2/4 - x^3/6)\Big|_{-4}^{3}$$

$$= (18 - 9/4 - 27/6) - (-24 - 4 + 64/6)$$

$$= 28.58$$

14. $f(x) = x^3$

$g(x) = x^2$

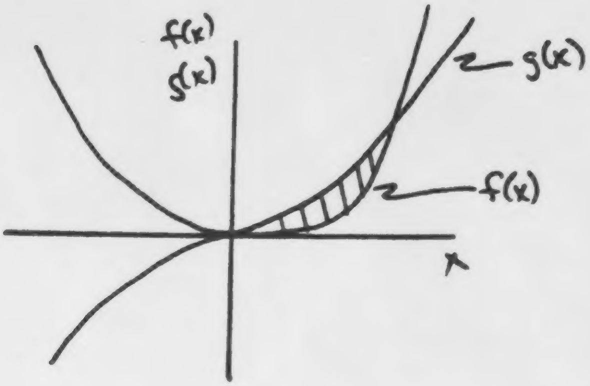
determining limits: $x^3 = x^2$

$$x^3 - x^2 = 0$$

$$x^2(x - 1) = 0$$

$$x = 0,\ 0,\ 1$$

sketching the function:



$$\text{Area} = \int_0^1 (x^2 - x^3)\,dx = (x^3/3 - x^4/4)\Big|_0^1 = 1/3 - 1/4 = 1/12$$

15. $f(x) = 1/x$ $(x > 0)$

$g(x) = 10 - x^2$

determining the limits: $1/x = 10 - x^2$

$$1 = 10x - x^3$$

$$x^3 - 10x + 1 = 0$$

$h(x) = x^3 - 10x + 1$

[This requires Newton's Method; refer to Study Guide, pp. 170-174.]

$$\begin{array}{r|rrrr} & 1 & 0 & -10 & 1 \\ \underline{0.1001} & & .1001 & +.1001 & -.99098 \\ \hline & 1 & .1001 & -9.8999 & 0 \end{array}$$

$(x - 0.1001)(x^2 + 0.1001x - 9.8999)$

$$x = \frac{-0.1001 \pm \sqrt{(0.1001)^2 - 4(1)(-9.8999)}}{2(1)}$$

$= -3.197, +3.097$

Thus, $x = +0.1001, +3.07, -3.197$

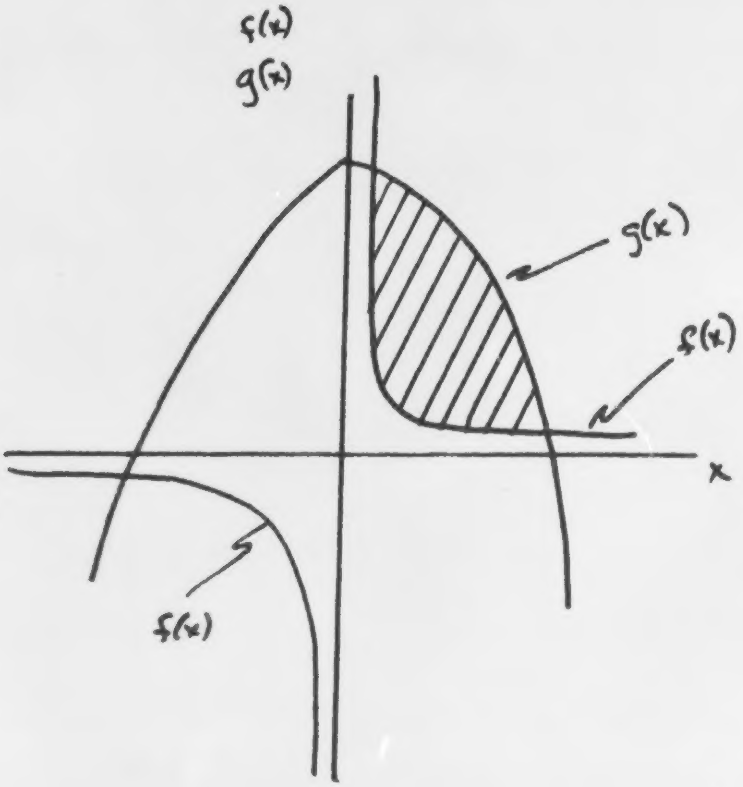
$h'(x) = 3x^2 - 10$

$h'(0.1001) = -$ (max)

$h'(3.097) = +$ (min)

$h''(-3.197) = +$ (min)

sketching the function:



$$\text{Area} = \int_{0.1001}^{3.097} [(10 - x^2) - 1/x]\, dx$$

$$= (10x - x^3/3 - \ln|x|) \Big|_{.1001}^{3.097}$$

$$= (30.97 - 9.90153 - 1.13043) - (1.001 - 0.00033 + 2.30159)$$

$$= 16.63578$$

16. $f(x) = 2(25 - x^2)$

$g(x) = 2 + 4x$

determining the limits:

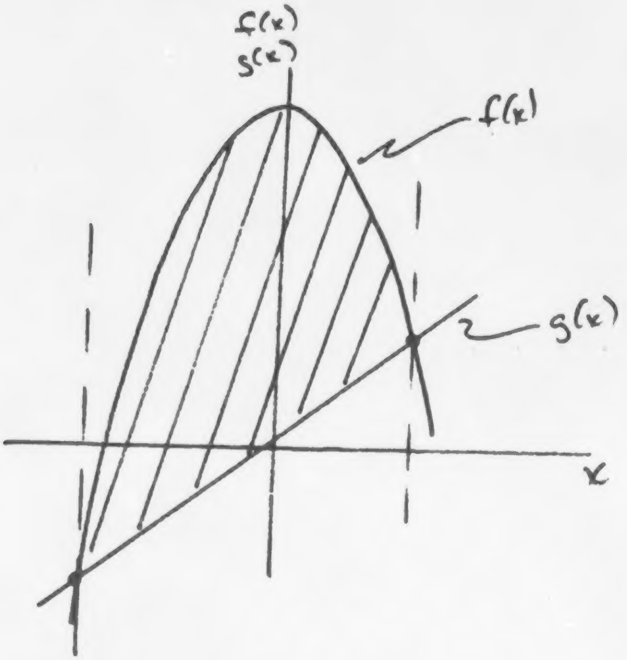
$$2(25 - x^2) = 2 + 4x$$

$$50 - 2x^2 = 2 + 4x$$

$$-2x^2 - 4x + 48 = 0$$

$$-2(x^2 + 2x - 24) = 0$$

$$x = -6,\ 4$$



$$\text{Area} = \int_{-6}^{4} [(50 - 2x^2) - (2 - 4x)]dx$$

$$= (48x - 2x^3/3 + 2x^2)\Big|_{-6}^{4}$$

$$= (192 - 128/3 + 32) - (-288 + 144 + 72)$$

$$= 253.33$$

17. $f(x) = 4x^2$

$g(x) = x^3$

determining the limits:

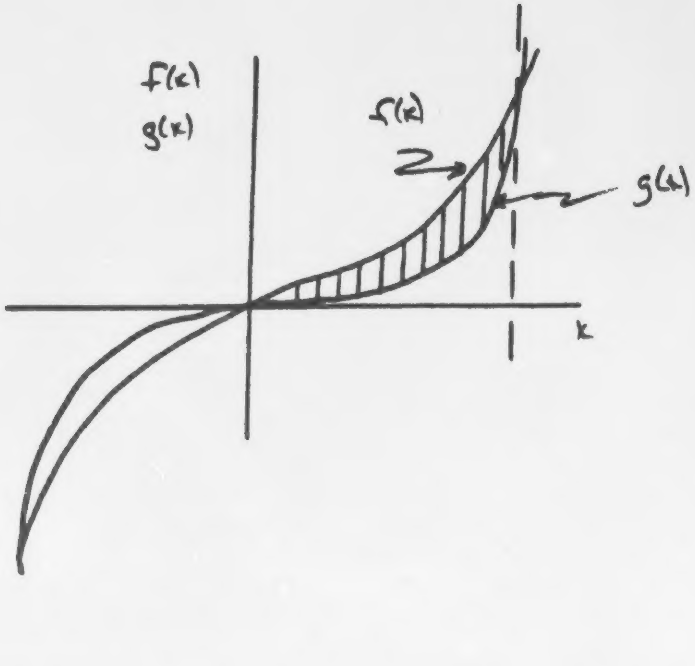
$$4x^2 = x^3$$

$$4x^2 - x^3 = 0$$

$$x^2(4 - x) = 0$$

$$x = 0, 0, 4$$

sketching the function:



$$\text{Area} = \int_{0}^{4} (4x^2 - x^3)\,dx$$

$$= (4x^3/3 - x^4/4)\Big|_{0}^{4}$$

$$= 256/3 - 256/4$$

$$= 21.33$$

18. $f(x) = x^2 - 1$

$g(x) = -x^2 + 5x - 3$

determining limits:

$x^2 - 1 = -x^2 + 5x - 3$

$2x^2 - 5x + 2 = 0$

$(2x - 1)(x - 2) = 0$

$x = 1/2,\ 2$

$f(1/2) = -3/4 \qquad g(1/2) = -1/4 + 5/2 - 3 = -3/4$

$f(2) = +3 \qquad g(2) = -4 + 10 - 3 = +3$

sketching the function:

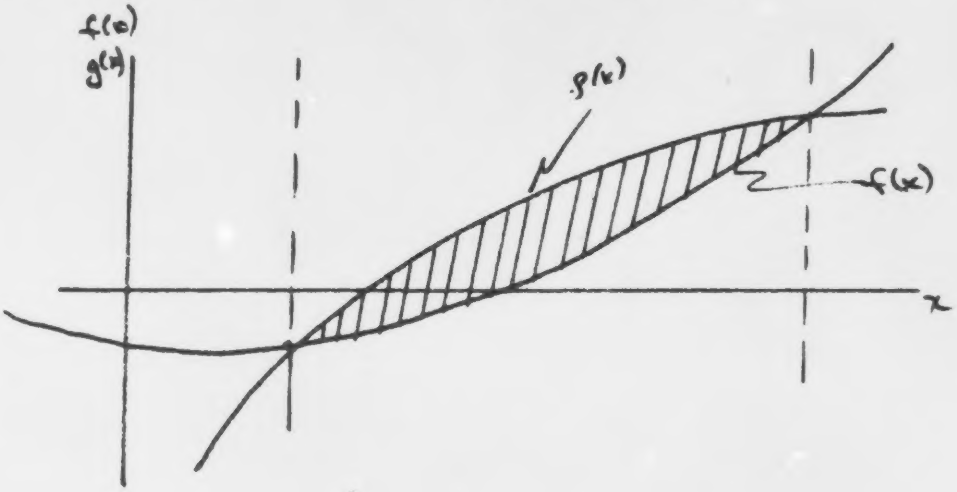
$f(x) = x^2 - 1 \qquad g(x) = -x^2 + 5x - 3$

$f'(x) = 2x \qquad g'(x) = -2x + 5$

$x^* = 0 \qquad x^* = 5/2$

$f(0) = -1 \qquad g(5/2) = 13/4$

$f''(x) = +2$ (min) $\qquad g''(x) = -2$ (max)



$$\text{Area} = \int_{1/2}^{2} [(-x^2 + 5x - 3) - (x^2 - 1)]\,dx$$

$$= (-2x^3/3 + 5x^2/2 - 2x)\Big|_{1/2}^{2}$$

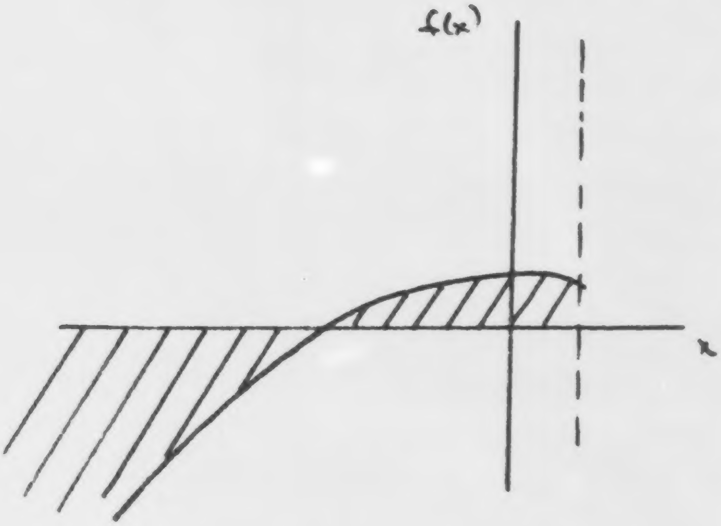
$$= [-2(8)/3 + 5(4)/2 - 2(2)] - [-2(1/8)/3 + 5(1/4)/2 - 2(1/2)]$$

$$= 9/8$$

Exercise 12.7, p. 496.

1. $\int_{-\infty}^{1} (10 - x^2)\,dx$

sketching the function:



$$-\int_{-\infty}^{-\sqrt{10}} (10 - x^2)\,dx + \int_{-\sqrt{10}}^{+1} (10 - x^2)\,dx$$

$$= (10x - x^3/3)\Big|_{-\infty}^{-\sqrt{10}} + (10x - x^3/3)\Big|_{-\sqrt{10}}^{+1}$$

$$= [(10\sqrt{10} - \frac{10\sqrt{10}}{3}) - (-\infty - \{\frac{-\infty}{3}\})]$$

$$+ [(10 - 1/3) - (-10\sqrt{10} + \frac{10\sqrt{10}}{3})]$$

275

$$= \frac{20\sqrt{10}}{3} - \infty + 20/3 + \frac{20\sqrt{10}}{3}$$

= is indeterminate

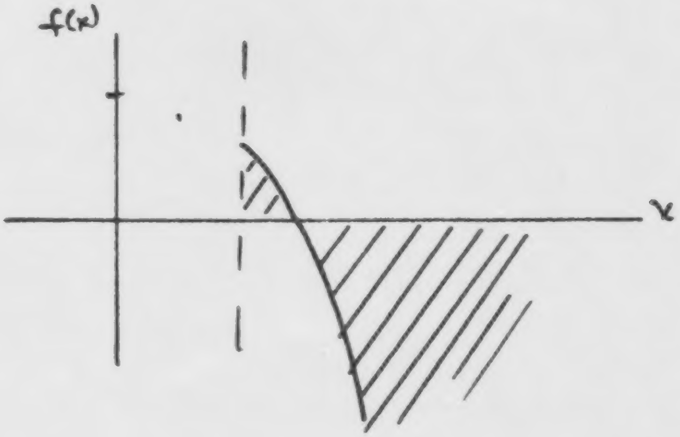
2. $$\int_{-\infty}^{+\infty} x^3\,dx = \int_{-\infty}^{0} x^3\,dx + \int_{0}^{+\infty} x^3\,dx$$

$$= x^4/4\,\Big|_{-\infty}^{0} + x^4/4\,\Big|_{0}^{+\infty}$$

$$= 0 - (-\infty/4) + (+\infty/4) - 0$$

= is indeterminate

3. $$\int_{2}^{+\infty} (3 + 2x - x^2)\,dx$$



sketching the function:

$f(x) = 3 + 2x - x^2$

$f'(x) = 2 - 2x$

$x^* = 1$

$f(1) = 3 + 2 - 1 = 4$

$$\int_{2}^{3}(3 + 2x - x^2)\,dx - \int_{3}^{+\infty}(3 + 2x - x^2)\,dx$$

$$= (3x + x^2 - x^3/3)\Big|_{2}^{3} - (3x + x^2 - x^3/3)\Big|_{3}^{+\infty}$$

$$= [(9 + 9 - 9) - (6 + 4 - 8/3)] - [\infty - (9 + 9 + 9)]$$

$$= 18 - 22/3 + \infty$$

= is indeterminate

4. $$\int_{1}^{\infty} 1/x\,dx = \ln|x|\Big|_{1}^{\infty} = \ln 1 - \ln \infty = \text{is indeterminate}$$

5. $$\int_{0}^{+\infty} e^x\,dx = e^x\Big|_{0}^{\infty} = 1 - e^{\infty} = \infty = \text{is indeterminate}$$

CHAPTER 13

INTEGRAL CALCULUS--ADVANCED TOPICS AND APPLICATIONS

Simple, but realistic applications make this chapter a must for those students who have mastered (or at least attempted to master) the basic concepts in Chapter 12.

Concepts of area are now applied in settings involving total profit. Rubbish removal sets the discussion mood for consumers' and producers' surplus. Financial processes are evaluated as an application of integral calculus.

The learning curve, a useful concept in its own right, develops the ideas of segments of area under a curve. This is expanded in the section on probability density functions.

The concept of mean, variance, and standard deviation is developed on a calculus base. This is a smooth transition, applying integral calculus when discussing probability. Most statistics professors mention that the foundation for discussing area under the normal curve is based on integral calculus. Here, in this chapter, the student is sequentially led into probability as a matter of normal progression. The stage is set for probability.

Exercise 13.6, pp. 518-520.

1. $R(x) = 50x - x^2$

$C(x) = 4x^2$

determining the limits:

$R(x) = C(x)$

$50x - x^2 = 4x^2$

$-5x^2 + 50x = 0$

$-5x(x - 10) = 0$

$x = 0, 10$

sketching the functions:

$R'(x) = 50 - 2x$

$0 = 50 - 2x$

$x^* = 25$

$R(25) = 50(25) - (25)^2 = 625$

$C'(x) = 8x$

$0 = 8x$

$x^* = 0$

$C(0) = 0$

$I = 20$

$C = 50(I)/Q + (.50/2)/Q^3 + 20I$

$C = 50(20)/Q + .25Q^3 + 400$

$$\frac{dC}{dQ} = \frac{-1000}{Q^2} + .75Q^2$$

$0 = -1000/Q^2 + .75Q^2$

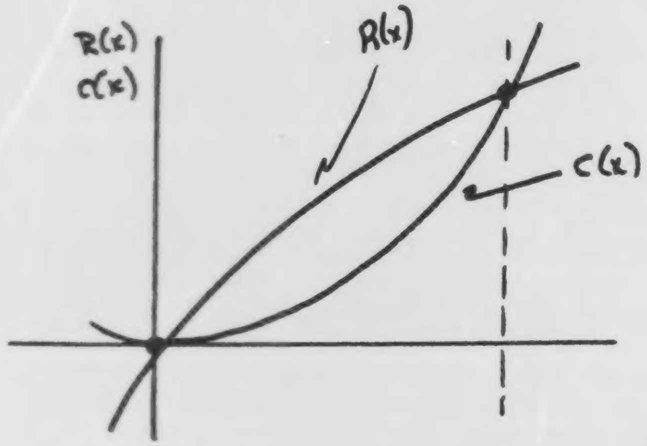
$Q^4 = 1000/.75 \qquad Q = 6.04$

$C = 50(20)/6.04 + .25(6.04)^3 + 400$

$= \$620.65$ per week

$$\frac{d^2C}{dQ^2} = \frac{2000}{Q^3} + 1.5Q > 0 \text{ always (minimum)}$$

sketching the functions:



$$P(x) = (50x - x^2) - (4x^2)$$

$$\int_0^{10} P(x) = \int_0^{10} (-5x^2 + 50x)\,dx = \frac{-5x^3}{3} + 25x^2 \Big|_0^{10}$$

$$= -1666.67 + 2500$$

$$= 833.33$$

2. $MC = x^2 - 16x + 20$

$MR = 20 - 2x$

$$MC = MR$$

$$x^2 - 16x + 20 = 20 - 2x$$

$$x^2 - 14x = 0$$

$$x = 0,\ 14$$

$$C = \int MC\,dx = \int (x^2 - 16x + 20)\,dx = \frac{x^3}{3} - 8x^2 + 20x$$

$$R = \int MR\,dx = \int (20 - 2x)\,dx = 20x - x^2$$

$$P(x) = (20x - x^2) - (x^3/3 - 8x^2 + 20x)$$

$$= x^3/x + 7x$$

$$P(14) = (14)^3/3 + 7(14)^2 = 2286.67$$

3. $D(q) = 25 - 5q + q^2/4$

$S(q) = 5q + q^2/4$

$$D(q) = S(q)$$

$$25 - 5q + q^2/4 = 5q + q^2/4$$

$$25 - 10q = 0$$

$$q = 2.5$$

$$S(q) = 5(2.5) + (2.5)^2/4 = 14.0625$$

$$PQ = 2.5(14.0625) = 35.15625$$

$$CS = \int_0^{2.5} (25 - 5q + q^2/4)\,dq - 35.15625$$

$$= (25q - 5q^2/2 + q^3/12)\Big|_0^{2.5} - 35.15625 = 13.02$$

$$PS = 35.15625 - \int_0^{2.5} (5q + q^2/4)\,dq$$

$$= 35.15625 - (5q^2/2 + q^3/12)\Big|_0^{2.5} = 18.23$$

4. $$\text{Total time} = 1354.5 \int_{100}^{150} x^{-0.7291}\,dx$$

$$= 5000x^{0.2709}\Big|_{100}^{150} = 2021.17 \text{ hours}$$

Total cost = 2021.17(\$15) = \$30,317.52

$$\text{Total time} = 1354.5 \int_{200}^{250} x^{-0.7291}\,dx$$

$$= 5000x^{0.2709}\Big|_{200}^{250} = 1308.90 \text{ hours}$$

Total cost = 1308.90(\$15) = \$19,633.43

The cost would be \$10,684.09 less.

5. $A = \int_0^5 1200\, e^{-(.07)t}\, dt$

$= \frac{1200}{-0.07}\, (e^{-.07(5)} - 1)$

$= \$5{,}062.49$

6. $A = \int_0^{30} 500\, e^{0.06t}\, dt$

$= \frac{500}{.06}(e^{0.06(30)} - 1)$

$= \$42{,}080.40$

7. $\frac{dp}{dq} = \frac{+1}{-4}$

$\frac{D(q) - 10}{q - 50} = -1/4$

$D(q) - 10 = +30/4 - 1/4q$

$D(q) = 90/4 - 1/4q$

8. $C(x) = 3x^2$

$S(x) = 2x^2 + 16$

determining limits: $C(x) = S(x)$

$3x^2 = 2x^2 + 16$

$x^2 = 16$

$x = \pm 4$

$x = 4$ units

$TS = \int_0^4 (2x^2 + 16)\, dx = (2x^3/3 + 16)\Big|_0^4 = \$106.67\times 10^3$

9. $R(x) = 16088e^{0.04x}$

day 1 $16088\int_{0}^{1} e^{0.04x}\,dx - \int_{0}^{1} 11.89dx$

$= 16088/.04(e^{0.04(1)} - 1) - 11.89(1)$

$= \$16,402.20$

day 5 $16088\int_{4}^{5} e^{0.04x}\,dx - \int_{4}^{5} 11.89$

$= [16088/.04(e^{0.04(x)} - 1) - 11.89x]\Big|_{4}^{5}$

$= \$88,988.74 - 69738.51 = \$19,250.23$

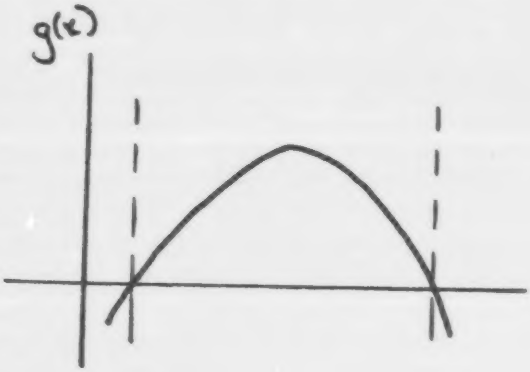
day 10 $16088 \int_{9}^{10} e^{0.04x}\,dx - \int_{9}^{10} 11.89$

$= [16088/.04\ (e^{0.04x} - 1) - 11.89x]\Big|_{9}^{10}$

$= \$197692.99 - 174178.08 = \$23,514.91$

10. $g(x) = -10 + 11x - x^2$ $(1 \le x \le 10$

sketching the function:



$g'(x) = 11- 2x$

$x^* = 11/2$

$g(1/2)= 81/4$

$$\int_{1}^{10} (-10 + 11x - x^2)\, dx$$

$$= (-10x + 11x^2/2 - x^3/3) \Big|_{1}^{10}$$

$$= (100 + 550 - 1000/3) - (-10 + 11/2 - 1/3)$$

$$= 243/2$$

$$f(x) = \frac{2(-10 + 11x - x^2)}{243} \qquad (1 \le x \le 10)$$

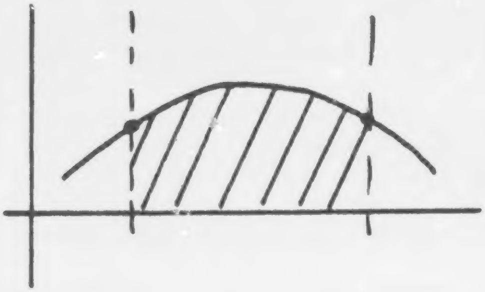
11. $g(x) = -5 + 6x - x^2 \qquad (2 \le x \le 4)$

sketching the function:

$g(x) = 6 - 2x$

$x^* = 3$

$f(3) = -5 + 18 - 9 = 4$



$$\int_{2}^{4} (-5 + 6x - x^2)dx = (-5x + 3x^2 - x^3/3) \Big|_{2}^{4}$$

$$= (-20 + 48 - 64/3) - (-10 + 12 - 8/3)$$

$$= 56/3$$

$$f(x) = \frac{3(-5 + 6x - x^2)}{56} \qquad (2 \le x \le 4)$$

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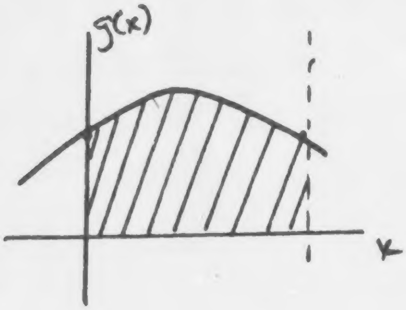
12. $g(x) = 4 + 3x - x^2$ $(0 \le x \le 3)$

sketching the function:

$g'(x) = 3 - 2x$

$x^* = 3/2$

$f(3/2) = 25/4$

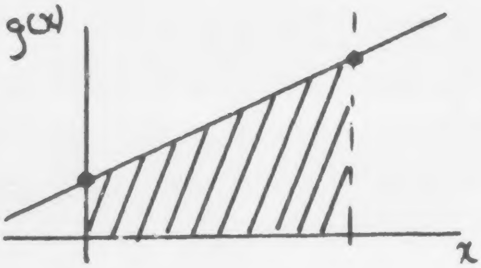


$$\int_0^3 (4 + 3x - x^2)dx = (4x + 3x^2/2 - x^3/3)\Big|_0^3$$

$$= 12 + 27/2 - 9 = 33/2$$

$f(x) = \dfrac{2(4 + 3x - x^2)}{33}$ $(0 \le x \le 3)$

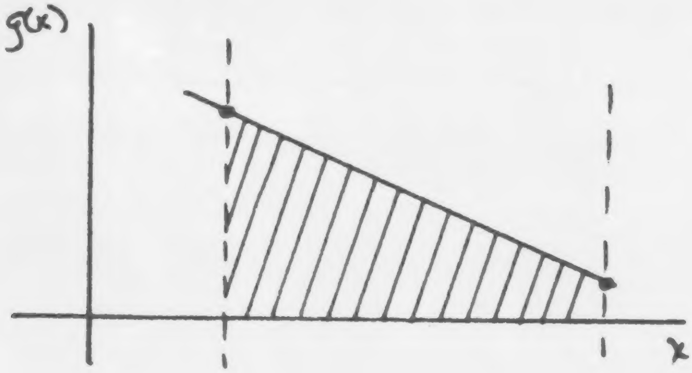
13. $g(x) = 2 + 1/2x$ $(0 \le x \le 8)$



$$\int_0^8 (2 + 1/2x)\ dx = 2x + x^2/4 \Big|_0^8 = 16 + 16 = 32$$

$f(x) = (2 + 1/2x)/32$ $(0 \le x \le 8)$

14. $g(x) = 10 - 1/4x$ $(8 \le x \le 32$



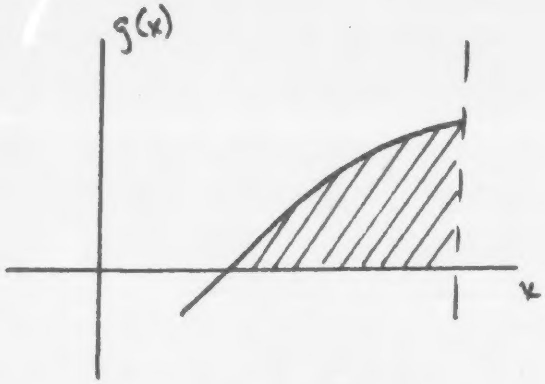
$$\int_{8}^{32} (10 - 1/4x)\,dx = (10x - x^2/8) \Big|_{8}^{32}$$

$$(320 - 128) - (80 - 8) = 120$$

$f(x) = (10 - 1/4)/120$ $(8 \le x \le 32)$

15. $g(x) = -75 + 20x - x^2$ $(5 \le x \le 10)$

sketching the function:



$g'(x) = 20 - 2x$

$x^* = 10$

$g(10) = 25$

$$\int_{5}^{10} (-75 + 20x - x^2)\,dx = (-75x + 10x^2 - x^3/3) \Big|_{5}^{10}$$

$$= (-750 + 1000 - 1000/3) - (-375 + 250 - 125/3)$$

$$= 83.333$$

$f(x) = (-75 + 20x - x^2)/83.333$ $(5 \le x \le 10)$

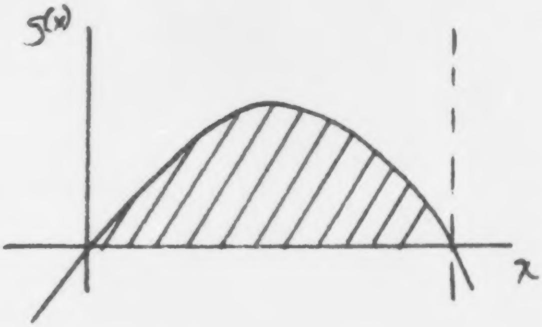
16. $g(x) = 2x + x^2 - x^3$ $(0 \le x \le 2)$

sketching the function:

$g'(x) = 2 + 2x - 3x^2$

$$x^* = \frac{-2 \pm \sqrt{4 - 4(-3)(2)}}{2(-3)} = +1.22, -0.55$$

$f(1.22) = 2.44 + 1.4884 - 1.8158 = 2.11$



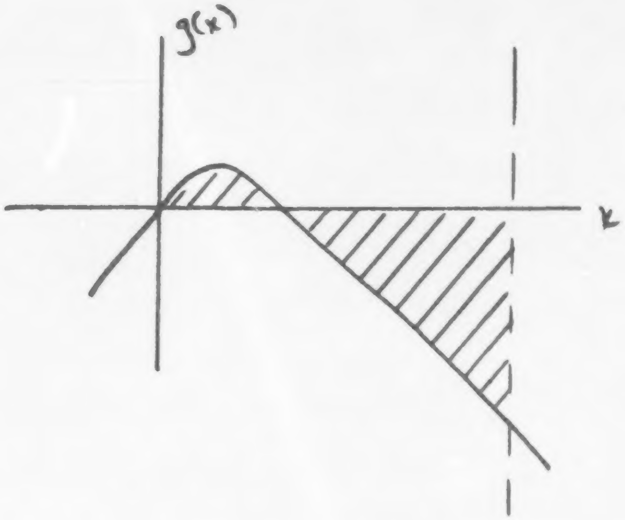
$$\int_0^2 (2x + x^2 - x^3)\, dx = (x^2 + x^3/3 - x^4/4) \Big|_0^2 = 8/3$$

$f(x) = 3(2x + x^2 - x^3)/8$ $(0 \le x \le 2)$

17. $g(x) = 5x(1 - x)$ $(0 \le x \le 5)$

$= 5x - 5x^2$

sketching the function:



$g'(x) = 5 - 10x$

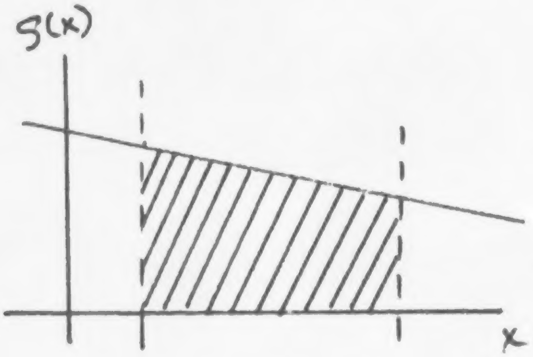
$x^* = 1/2$

$g(1/2) = 5/4$

violates assumuption 2

(see page 512)

18. $g(x) = 3 - 0.1x$ $(2 \le x \le 10)$

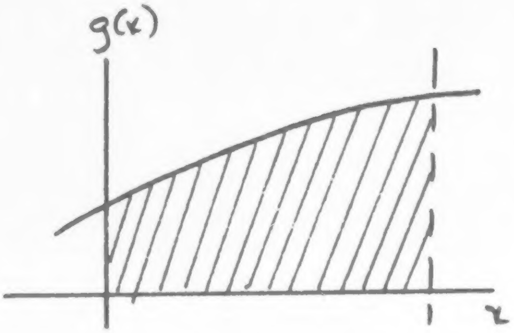


$$\int_{2}^{10} (3 - 0.1x)\,dx = (3x = 0.05x^2) \Big|_{2}^{10}$$

$$= (30 - 5) - (6 - 0.2) = 19.2 = 96/5$$

$f(x) = 5(3 - 0.1x)/96$ $(2 \le x \le 10)$

19. $g(x) = 4 + x^{1/2}$ $(0 \le x \le 5)$



$$\int_{0}^{5} (4 + x^{1/2})\,dx = (4x + 2x^{3/2}) \Big|_{0}^{5} = 20 + 22.36 = 42.36$$

$f(x) = (4 + x^{1/2})/42.36$ $(0 \le x \le 5)$

NOTE: Problems 20, 21, 22, and 23 are demonstrating the components making up the area contained in $(2 \le x \le 6)$. The functions for all four problems are variations of

$$f(x) = \frac{3(-12 + 8x - x^2)}{32} \qquad (2 \le x \le 6)$$

20. $$\text{Area} = \int_2^3 \frac{3(-12 + 8x - x^2)}{32}\, dx$$

$$= (3/32)(-12x + 4x^2 - x^3/3) \Big|_2^3$$

$$= (3/32)[(-36 + 36 - 9) - (-24 + 16 - 8/3)]$$

$$= (3/32)(1.6667) = 0.1562$$

21. $$\text{Area} = (9/1024) \int_3^4 (-128 + 256x/3 - 32x^2/3)\, dx$$

$$= (9/1024)[(-128x + 256x^2/6 - 32x^3/9) \Big|_3^4]$$

$$= (9/1024)[(-512 + 4096/6 - 2048/9) - (-384 + 384 - 96)]$$

$$= 0.3438$$

22. $$\text{Area} = (3/32) \int_4^5 (-12 + 8x - x^2)\, dx$$

$$= (3/32)(-12x + 4x^2 - x^3/3) \Big|_4^5$$

$$= (3/32)[(-60 + 100 - 125/3) - (-48 + 64 - 64/3)]$$

$$= 0.3438$$

23. $$\text{Area} = (9/1024) \int_5^6 (-128 + 85.33x - 10.67x^2)\, dx$$

$$= (9/1024)[(-128x + 85.33x^2/2 - 10.67x^3/3) \Big|_5^6]$$

$$= (9/1024)[(-768 + 1535.94 - 768.24) - (-640 + 1066.625$$

$$- 444.583)]$$

$$= 0.1552$$

24. $$\text{Area} = \int_0^3 (1/4 - x^2/36)\,dx$$

$$= (x/4 - x^3/108)\Big|_0^3 = 3/4 - 27/108 = 0.5000$$

25. $$\text{Area} = \int_0^{10} (0.015x - 0.00075x^2)\,dx$$

$$= (0.015x^2/2 - 0.00025x^3)\Big|_0^{10}$$

$$= 0.75 - 0.25 = 0.5000$$

26. $$\text{Area} = (1/36)\int_2^5 (-7 + 8x - x^2)\,dx$$

$$= (1/36)(-7x + 4x^2 - x^3/3)\Big|_2^5$$

$$= (1/36)[(-36 + 100 - 125/3) - (-14 + 16 - 8/3)]$$

$$= 0.6389$$

27. $$g(x) = -10 + 7x - x^2$$

$$0 = -10 + 7x - x^2$$

$$0 = (-5 + x)(2 - x)$$

$$x = 2,\ 5$$

$$g(x) = \int_2^5 (-10 + 7x - x^2)\,dx$$

$$= (-10x + 7x^2/2 - x^3/3)\Big|_2^5$$

$$= (-50 + 175/2 - 125/3) - (-20 + 14 - 8/3) = 4.5$$

Thus, $f(x) = \dfrac{2(-10 + 7x - x^2)}{9} \qquad (2 \le x \le 5)$

$$\text{Area} = (2/9)\int_2^3 (-10 + 7x - x^2)\,dx = (2/9)(-10x + 7x^2/2 - x^3/3)\Big|_2^3$$

$$= (2/9)[(-30 + 63/2 - 9) - (-20 + 14 - 8/3)] = 0.2593$$

28. $g(x) = -12 + 8x - x^2 \qquad (2 \le x \le 5)$

$$0 = -12 + 8x - x^2$$

$$0 = (-6 + x)(2 - x)$$

$$x = 2,\ 6$$

$$g(x) = \int_2^6 (-12 + 8x - x^2)\,dx = (-12x + 4x^2 - x^3/3)\Big|_2^6$$

$$= (-72 + 144 - 72) - (-24 + 16 - 8/3)$$

$$= 10.6667 = 32/3$$

Thus, $f(x) = \dfrac{3(-12 + 8x - x^2)}{32} \qquad (2 \le x \le 6)$

(This is the p.d.f. used for problems 20-23.)

$$\text{Area} = (3/32)\int_2^5 (-12 + 8x - x^2)\,dx$$

$$= (3/32)(-12x + 4x^2 - x^3/3)\Big|_2^5$$

$$= (3/32)[(-60 + 100 - 125/3) - (-24 + 16 - 8/3)]$$

$$= 0.8438$$

check:

$$\int_2^5 f(x)dx = \int_2^3 f(x)\,dx + \int_3^4 f(x)\,dx + \int_4^5 f(x)\,dx$$

$$0.8438 = 0.1562 + 0.3438 + 0.3438$$

29. $g(x) = 20 + x - x^2$

$$0 = 20 + x - x^2$$

$$0 = (5 - x)(4 + x)$$

$$x = -4,\ 5$$

$$G(x) = \int_{-4}^5 (20 + x - x^2)\,dx = (20x + x^2/2 - x^3/3)\Big|_{-4}^5$$

$$= (100 + 25/2 - 125/3) - (-80 + 8 + 64/3)$$

$$= 121.5 = 243/2$$

Thus,

$$f(x) = \frac{2(20 + x - x^2)}{243} \qquad (-4 \le x \le 5)$$

$$\text{Area} = (2/243)\int_0^2 (20 + x - x^2)\,dx = (2/243)(20x + x^2/2 - x^3/3)\Big|_0^2$$

$$= (2/243)(40 + 2 - 8/3) = 0.3237$$

30. $g(x) = 5x - 5x^2 \qquad (0 \le x \le 5)$

(Refer to problem 17 in this section; g(x) violates assumption 2 on page 512.)

31. $g(x) = 4 + 3x - x^2 \qquad (0 \le x \le 3)$

$0 = (4 - x)(1 + x)$

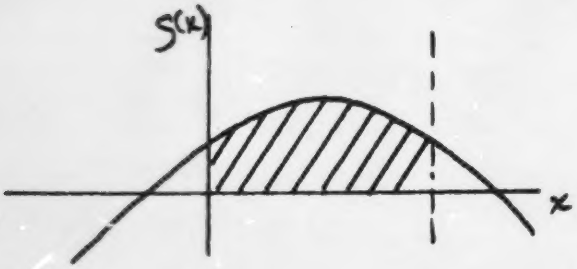
$x = -1, 4$

skething the function:

$g'(x) = 3 - 2x$

$x^* = 3/2$

$g(3/2) = 4 + 9/2 - 9/4 = 25/4$



$$G(x) = \int_0^3 (4 + 3x - x^2)\,dx = (4x + 3x^2/2 - x^3/3)\Big|_0^3 = 33/2$$

Thus,

$$f(x) = \frac{2(4 + 3x - x^2)}{33} \qquad (0 \le x \le 3)$$

$$\mu_x = \int_0^3 x(f(x)\,dx = (2/33)\int_0^3 x(4 + 3x - x^2)\,dx$$

$$= (2/33)\int_0^3 (4x + 3x^2 - x^3)\,dx = (2/33)(2x^2 + x^3 - x^4/4)_0^3 = 1.5$$

$$\sigma_x^2 = \int_0^3 (x - \mu_x)^2 f(x)\, dx$$

$$= (2/33)\int_0^3 (x - 1.5)^2 (4 + 3x - x^2)\, dx$$

$$= (2/33)\int_0^3 (x^2 - 3x + 2.25)(4 + 3x - x^2)\, dx$$

$$= (2/33)\int_0^3 (-x^4 + 6x^3 - 7.25x^2 - 5.25x + 9)\, dx$$

$$= (2/33)(-x^5/5 + 6x^4/4 - 7.25x^3/3 - 5.25x^2/2 + 9x)\Big|_0^3$$

$$= (2/33)(-48.6 + 121.5 - 62.25 - 23.625 + 27)$$

$$\sigma_x^2 = 0.67$$

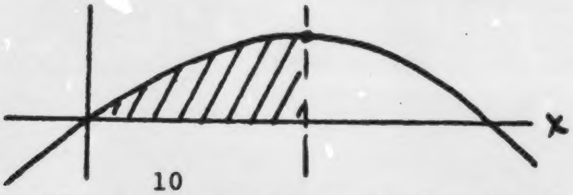
$$\sigma_x = \sqrt{0.67} = 0.82$$

32. $g(x) = 0.015x - 0.00075x^2$ $\quad (0 \le x \le 10)$

$$0 = 0.015x(1 - 0.05x)$$

$$x = 0,\ 20$$

sketching the function:



$$g'(x) = 0.015 - 0.0015x$$

$$x^* = 10$$

$$g(10) = 0.075$$

$$G(x) = \int_0^{10} (0.015x - 0.00075x^2)\, dx = 0.5000$$ (Refer to problem 25 in this section.)

Thus,

$$f(x) = 2(0.015x - 0.00075x^2) \quad (0 \le x \le 10)$$

$$= 0.03x - 0.0015x^2$$

$$\mu_x = \int_0^{10} x\, f(x)\, dx = \int_0^{10} (0.03x^2 - 0.0015x^3)\, dx$$

$$= (0.01x^3 - 0.000375x^4)\Big|_0^{10} = 6.25$$

$$\sigma_x^2 = \int_0^{10} (x - 6.25)^2 (0.03x - 0.0015x^2)\, dx$$

$$= \int_0^{10} (x^2 - 12.5 + 39.0625)(0.03x - 0.0015x^2)\, dx$$

$$= \int_0^{10} (-0.0015x^4 + 0.04875x^3 - 0.43359x^2 + 1.171875x)\, dx$$

$$= (-0.0003x^5 + 0.01219x^4 - 0.14453x^3 + 0.58594x^2) \Big|_0^{10}$$

$$= (-30 + 121.9 - 144.53 + 58.594) = 5.96$$

$$\sigma_x = \sqrt{5.96} = 2.44$$

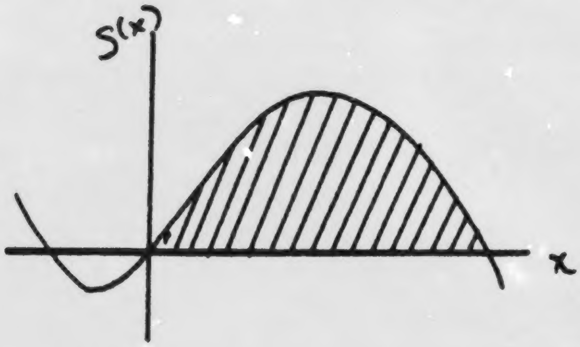
33. $g(x) = 2x + x^2 - x^3$

$0 = x(2 + x - x^2)$

$0 = x(2 - x)(1 + x)$

$x = -1, 0, 2$

sketching the function:



$g'(x) = 2 + 2x - 3x^2$

$0 = 2 + 2x - 3x^2$

$x^* = 1.18, -0.52$

$g''(x) = 2 - 6x$

$g''(1.18) = -5.08$ (max)

$g''(-0.52) = +5.12$ (min)

$g(1.18) = 2.11$

$g(-.52) = -0.63$

$$G(x) = \int_0^2 (2x + x^2 - x^3) = (x^2 + x^3/3 - x^4/4)\Big|_0^2 = 2.67 = 8/3$$

Thus, $f(x) = \dfrac{3(2x + x^2 - x^3)}{8} \qquad (0 \le x \le 2)$

$$\mu_x = \int_0^2 x\, f(x)\, dx = (3/8)\int_0^2 (2x^2 + x^3 - x^4)\, dx$$

$$= (3/8)(2x^3/3 + x^4/4 - x^5/5)\Big|_0^2 = 1.1$$

$$\sigma_x^2 = (3/8)\int_0^2 (x - 1.1)^2(2x + x^2 - x^3)\, dx$$

$$= (3/8)\int_0^2 (x^2 - 2.2x + 1.21)(2x + x^2 - x^3)\, dx$$

$$= (3/8)\int_0^2 (-x^5 + 3.2x^4 - 1.41x^3 - 3.21x^2 + 2.42x)\, dx$$

$$= (3/8)\ (-x^6/6 + 0.64x^5 - 0.285x^4 - 1.07x^3 + 1.21x^2)\Big|_0^2$$

$$= (3/8)(-10.67 + 20.48 - 4.56 - 8.56 + 4.84) = 0.57$$

$$\sigma_x = \sqrt{0.57} = 0.75$$

34. $g(x) = -10 + 11x - x^2$ $\qquad (1 \le x \le 10)$

(see sketch in problem 10 of this section)

$$f(x) = \frac{2(-10 + 11x - x^2)}{243} \qquad (1 \le x \le 10)$$

$$\mu_x = \int_1^{10} x\, f(x)\, dx = (2/243)\int_1^{10} (-10x + 11x^2 - x^3)\, dx$$

$$= (2/243)(-5x^2 + 11x^3/3 - x^4/4)\Big|_1^{10}$$

$$= (2/243)[(-500 + 3666.67 - 2500) - (-5 + 11/3 - 1/4)]$$

$$= 5.5$$

$$\sigma_x^2 = (2/243) \int_1^{10} (x - 5.5)^2 (-10 + 11x - x^2)\ dx$$

$$= (2/243) \int_1^{10} (-x^4 + 22x^3 - 161.25x^2 + 442.75x - 302.5)\ dx$$

$$= (2/243)(-0.2x^5 + 5.5x^4 - 53.75x^3 + 221.375x^2 - 302.5x)\ \Big|_1^{10}$$

$$= (2/243)[(-20{,}000 + 55{,}000 - 53{,}750 + 22{,}137.5 - 3025)$$

$$- (-0.2 + 5.5 - 53.75 + 221.375 - 302.5)]$$

$$= 4.05$$

$$= \sqrt{4.05} = 2.01$$

CHAPTER 14

PROBABILITY THEORY

Basic probability theory is introduced in this chapter. Most of the material was presented initially in sections 0.2 and 0.3. Factorials, permutations, and combinations are discussed.

This chapter will be beneficial to those students enrolling in a basic statistics course.

1. (a) Since $P(A \cap B) \neq 0$, events are NOT mutually exclusive.

   (b) Since $P(A)P(B) \neq P(A \cap B)$, that is $(0.3)(0.6) \neq (0.5)$, the events are NOT independent.

2. The information could be displayed as:

	B_1	B_2	B_3	
A_1	1/5			3/5
A_2		1/3		1/5
A_3			2/5	1/5
	1/3	1/6	1/2	(1)

(a) $P(\bar{A}_1 \cap B_1) = P(A_2 \cap B_1) + P(A_3 \cap B_1)$

$= P(B_1) - P(A_1 \cap B_1)$

$= (1/3) - (1/5)$

$= 5/15 - 3/15$

$= 2/15$

$P(\bar{A}_2 \cap B_2) = P(B_2) - P(A_2 \cap B_2)$

$= (1/6) - (1/3)$

$= 1/3$

$P(\bar{A}_3 \cap B_3) = P(B_3) - P(A_3 \cap B_3)$

$= (1/2) - (2/5)$

$= 1/10$

(b) Not known; not enough information

(c) No, $P(A_i)P(B_i) \neq P(A_i \cap B_i)$ $(i = 1, 2, 3)$

3. (a) 0.40

(b) 0.10

(c) 0.05

(d) 0.02

(e) $0.20 + 0.13 - 0.05 = 0.28$

(f) $\emptyset$ (impossible event)

(g) 0 (mutually exclusive)

(h) $0.05/0.19 = 0.26$

(i) $0.02/0.10 = 0.20$

(j) $A_1 \cap B_3$, $A_1 \cap B_5$

(k) No, $P(A_i)P(B_i) \neq P(A_i \cap B_i)$

4. (a) $300/400 = 0.7500$ $P(D/F)$

(b) $75/200 = 0.3750$ $P(O/R)$

(c) $25/800 = 0.03125$ $P(N \cap R)$

$75/800 = 0.09375$ $P(N \cap D)$

(d) No, $P(R)P(\text{Opinion}) \neq P(R \cap O)$

$P(D)P(\text{Opinion}) \neq P(D \cap O)$

5.

	B_1	B_2	B_3	B_4	
A_1	.15	.26	.09	.20	.70
A_2	.15	.04	.11	0	.30
	.30	.30	.20	.20	1.00

Given: $P(A_2 \cup B_3) = 0.39$, then using

$P(A_2) + P(B_3) - P(A_2 \cap B_3)$,

$.30 + .20 - x = 0.39$

$x = 0.11$

Thus, $P(A_2 \cap B_3) = x = 0.11$

6. A realizes that B and C are the biggest threat to each other. Thus, A does not desire to compete with B or C, since A would most likely be outbid by the survivor. By A's not bidding, B and C will concentrate on each other. This will then permit A to bid against the survivor of B and C. [This is a logic

question demonstrating an intuitive approach to an almost impossible setting.]

7. (a) $P(A_1 \cup B_1)$

$$P(A_1 \cup B_1) = P(A_1) + P(B_1) - P(A_1 \cap B_1)$$

$$= \left(\frac{2346}{5026}\right) + \left(\frac{1532}{5026}\right) - \left(\frac{704}{5026}\right)$$

$$= 0.6315$$

(b) $P(A_2 \cup A_4) = \emptyset$ (given that a single major must be declared)

(c) $P(A_4 \cap B_2)$

$$P(A_4 \cap B_2) = P(A_4/B_2)P(B_2)$$

$$= \left(\frac{449}{3494}\right)\left(\frac{3494}{5026}\right)$$

$$= 0.0893$$

also,

$$P(A_4 \cap B_2) = P(B_2/A_4)P(A_4)$$

$$= \left(\frac{449}{659}\right)\left(\frac{659}{5026}\right)$$

$$= 0.0893$$

(d) $P(A_4/B_1)$

$$P(A_4/B_1) = P(A_4 \cap B_1)/P(B_1)$$

$$= \left(\frac{210}{5026}\right) \Big/ \left(\frac{1532}{5026}\right)$$

$$= (210/1532)$$

$$= 0.1371$$

(e) $P(B_1/A_4)$

$$P(B_1/A_4) = P(B_1 \cap A_4)/P(A_4)$$

$$= \left(\frac{210}{5026}\right) \Big/ \left(\frac{656}{5026}\right)$$

$$= (210/656)$$

$$= 0.3201$$

(f) $P(A_6 \cup B_2)$

$$P(A_6 \cup B_2) = P(A_6) + P(B_2) - P(A_6 \cap B_2)$$

$$= \frac{146}{5026} + \frac{3494}{5026} - \frac{66}{5026}$$

$$= 0.7111$$

Exercise 14.4, p. 540.

1. $2^{10} = 1024$

2. $6^5 = 7776$

3. $4^6 = 4096$

4. $5^7 = 78125$

5. ${}_{20}P_5 = \frac{20!}{(20 - 5)!} = 1{,}860{,}480$

6. ${}_{15}C_6 = \frac{15!}{6!(15 - 6)!} = 5005$

7. ${}_{7}P_3 = \frac{7!}{(7 - 3)!} = 210$

8. If order is important (and it frequently is important on grocery store shelves):

$${}_{12}P_8 = \frac{12!}{(12 - 8)!} = 19{,}958{,}400$$

If order is not important:

$${}_{12}C_8 = \frac{12!}{8!(12 - 8)!} = 495$$

If different sequences were of concern:

$$8^{12} = 6.87 \times 10^{10}$$

9. Order is not important, thus

$${}_{20}C_5 = \frac{20!}{5!(20 - 5)!} = 15{,}504$$

10. There are 26 letters and ten digits (0-9), thus

$$(26!\ 26!\ 26!)(10!\ 10!\ 10!) - 57{,}290$$

$$= 3.134340533 \times 10^{99}$$

(In 1980, California introduced a regular issue license plate consisting of a digit, three letters, and three numbers.

This will provide $10!(26!)^3(10!)^3 - x)$

CHAPTER 15

PROBABILITY DISTRIBUTIONS

The binomial, poisson, and normal distributions are introduced in this chapter. Tables in the appendix will facilitate the problem solving necessary to work the exercises.

The presentation of probability distributions builds on Chapters 12, 13, and 14. The final section deals with transforming distributions. Example 18 addresses a question I have been asked more times than I'd like to remember. Being a sailing enthusiast, I am asked, "Why didn't the first sailboat to finish win?" Have you ever tried to discuss the concept of handicap intelligently?

Your students should now have the necessary quantitative skills to succeed in their academic program.

Exercise 15.2, pp. 546-548.

Note: Binomial Distribution, Appendix J, may be used.

1. $n = 5$, $p(\text{defective}) = 0.2$

2. $P(x = 2) = \binom{5}{2}(0.2)^2(0.8)^3$

$$= \frac{5!}{2!(5-2)!}\ (0.2)^2(0.8)^3$$

$$= 120/12(.04)(.512)$$

$P(x = 2) = 0.2048$

(b) $P(x > 1) = 1 - P(x \le 1)$ (where $x > 1 = x \ge 2$)

$= 1 - [P(0) + P(1)]$

$= 1 - (0.3277 + 0.4096)$

$P(x > 1) = 0.2627$

(c) $P(x > 0) = 1 - P(x \le 0)$ (where $x > 0 = x \ge 1$)

$= 1 - P(0)$

$= 1 - 0.3277$

$P(x > 0) = 0.6723$

(d) $P(x = 0) = \binom{5}{0}(0.2)^0(0.8)^5$

$= \frac{5!}{0!(5-0)!} \quad (0.2)^0(0.8)^5$

$= (120/120)(1)(.3277)$

$P(x = 0) = .3277$

2. $n = 19$, $p(\text{ill}) = 0.05$

(a) $P(x = 3) = 0.0533$

(b) $P(x = 0) = 0.3774$

(c) $P(x > 1) = 1 - P(x \le 1)$ (where $x > 1 = x \ge 2$)

$= 1 - [P(0) + P(1)]$

$= 1 - (.3774 + 0.3774)$

$P(X > 1) = 0.2452$

(d) $P(x \le 5) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$

$= 0.3774 + 0.3774 + 0.1787 + 0.0533 + 0.0112 + 0.0018$

$P(x \le 5) = 0.9999$

[If $1 - P(x > 5)$ is used, then $1 - 0.0002 = 0.9998$, with the difference due to rounding errors in table]

(e) $P(2 \le x \le 5) = 1 - [P(x < 1) + P(x > 5)]$

$= 1 - (0.7548 + 0.0002)$

$= 0.2450$

[If used $P(2 \le x \le 5) = P(2) + P(3) + P(4) + P(5)$

then value $= 0.1787 + 0.0533 + 0.0112 + 0.0018$

$= 0.2450$

3. $n = 10$, $p(\text{error}) = 0.02$, $x = 4$

$P(x = 4) = \binom{10}{4}(0.02)^4(.098)^6$

$= \frac{10!}{4!(10-4)!}(0.02)^4(0.98)^6$

$P(x = 4) = 0.00003$

4. $n = 14,\ p(\text{sun}) = 0.70,\ x = 5$

$$\binom{14}{5}(0.70)^5(0.30)^9 = \binom{14}{9}(0.30)^9(0.70)^5 = 0.0066$$

5. $P(x) = 0.5$, postal service, time lead, responsiveness, etc.

6. $n = 6,\ p(\text{women}) = 15/50 = 0.3,\ p(\text{men}) = 35/50 = 0.7$

Let x represent number of women in group

(a) $P(x = 5) = \binom{6}{6}(0.3)^6(0.7)^0 = 0.0007$

(b) $P(x = 0) = \binom{6}{0}(0.3)^0(0.7)^6 = 0.1176$

(c) $P(x > 1) = 1 - P(x \le 1)$ (where $x > 1 = x \ge 2$)

$= 1 - [P(0) + P(1)]$

$= 1 - (0.1176 + 0.3025)$

$P(X > 1) = 0.5799$

(d) $P(n - x) = P(1) = \binom{6}{1}(0.7)^1(0.3)^5$

$= \binom{6}{5}(0.3)^5(0.7)^1 = 0.0102$

(e) $P(x \le 4) = 1 - [P(5) + P(0)$

$= 1 - (0.0102 + 0.0007)$

$= 0.9891$

7. $n = 4,\ p(\text{smokers}) = 0.40$

(a) $P(x = 0) = 0.1296$

(b) $P(x > 0) = 1 - P(0)$

$= 1 - .1296$

$P(x > 0) = 0.8704$

(c) $P(x = 4) = 0.0246$

(d) (now $n = 16$) $P(x = 0) = \binom{16}{0}(0.4)^0(0.6)^{16} = 0.0003$

8. Number of seats left = 52,000 - 20,000 = 32,000

Number of good orders = 73,257(.90) = 65,931.3

Number of tickets in good orders = 3.9(65,931.3) = 257,132.07

And, probability of getting a good order equals average number of orders selected divided by total number of good orders. Thus

$$P = (32,000/3.9)/65,931.3 = 0.1244$$

(a) Probability of at least one order's being picked is
1 - P(none being picked)

$$P = 1 \binom{n}{0} (0.1244)^0 (0.8756)^n$$

$$= 1 - (.8756)^n$$

n	P
1	0.1244
2	0.2333
3	0.3287
4	0.4122
5	0.4853
6	0.5494
7	0.6054
8	0.6545
9	0.6975
10	0.7351
11	0.7681
12	0.7969
13	0.8222
14	0.8443
15	0.8637
16	0.8806
17	0.8955
18	0.9085
19	0.9199
20	0.9298

check:

$$(1 - .1244)^n = .10$$

$$.8756^n = .10$$

$$n = \log_{.8756} 0.10$$

$$= \frac{\log 0.10}{\log . 0.8756} = 17.33 = 18$$

18 orders result in 90% chance of being picked

(b) Total of six orders sent

$P(x \geq 1) = 1 - P(x = 0)$

$= 1 - \binom{6}{0}(.1244)^0(.8756)^6$

$= 0.5494$

$P(x = 2) = \binom{6}{2}(.1244)^2(.8756)^4$

$= 0.1364$

(c) Total of three orders received

$P(x = 3) = \binom{6}{3}(.1244)^3(.8756)^3$

$= 0.0258$

Exercise 15.3, pp. 551-552.

Note: Poisson Distribution, Appendix K, may be used.

1. (a) $P(x = 10) = \frac{10^{10}e^{-10}}{10!}$ [With $\mu = 10$, this is usually written $P(x|\mu)$ where $P(x|\mu) = P(10|10)$.]

$P(x = 10) = 0.1251$

(b) $P(x < 9) = P(0) + P(1) + . . . + P(8)$

$= 0.3329$

[Could use $P(x < 9|10)$]

(c) $P(x > 2) = 1 - P(x \leq 2)$

$= 1 - [P(0) + P(1) + P(2)]$

$= 1 - [0 + o.0005 + 0.0023]$

$P(x > 2) = 0.9972$

(d) $P(x = 0) = 0.0000$

(d) $P(3 \leq x \leq 8) = P(3) + p(4) + . . . + P(8)$

$= 0.3301$

2. $\mu = 1$

$P(x \geq 3) = 1 - [P(0) + P(1) + P(2)]$

$= 1 - [0.3679 + 0.3679 + 0.1839]$

$P(x \geq 3) = 0.0803$

3. $\mu = 4$ (weekly)

$\mu = 8$ (bi-weekly)

$\mu = 16$ monthly

$P(x = 6 | \mu = 8) = 0.1221$

$P(x = 6 | \nu = 16) = 0.0026$

4. Since distribution is binomial, assume 50-50 chance of being delayed. Thus, $x = 25$ will be delayed.

$\mu = np = 50(.0.02) = 1$

$$P(25) = \frac{1^{25}e^{-1}}{25!}$$

$= 0.0000$

5. (a) $\mu = 2.4$, $P(x = 0) = 0.0907$

(b) $\mu = 2.4$, $P(x = 1) = 0.2177$

(c) $\mu = 0.8$, $P(x = 1) = 0.3595$

(d) $\mu = 0.8$, $P(x = 2) = 0.1438$

(e) $\mu = 2.4$, $P(x = 3) = 0.2090$

$\mu = 0.8$, $p(x = 1)^3 = (0.3595)^3 = 0.0465$; there are many ways of getting three sales, only one of each salesperson getting one sale.

6. Given: one galley page equals 2.5 text pages. Thus, two galley pages equals 5 text pages. If $\mu = 3$ for 2 galley pages, the μ for text pages equals $3/5 = 0.60$. Consequently, for galley pages, $\mu = 1.5$ and for text pages, $\mu = 0.60$.

(a) $\mu = 1.5$, $P(x = 0) = 0.2231$

(b) $\mu = 0.6$, $P(x = 0) = 0.5488$

(c) $\mu = 1.5$, $P(x = 3) = 0.1255$

(d) $\mu = 0.6$, $P(x = 2) = 0.0988$

(e) $\mu = (0.6)(10) = 6$

$$P(x < 5) = 0.0025 + 0.0149 + 0.0446 + 0.0892 + 0.1339$$
$$= 0.2851$$

Exercise 15.4, p. 557.

1. $P(0 \le z \le 2.02) = 0.4783$
2. $P(0 \le z \le 3.01) = 0.4987$
3. $P(0 \le z \le 0.56) = 0.2123$
4. $P(-1.2 \le z \le 0) = 0.3849$
5. $P(-2.58 \le z \le 0) = 0.4951$
6. $P(-1 \le z \le 1) = (0.3413 + 0.3413) = 0.6826$
7. $P(+2.45 \le z \le 2.9) = (0.4981 - 0.4929) = 0.0052$
8. $P(-2 \le z \le +2) = (0.4772 + 0.4772) = 0.9544$
9. $P(-.02 \le z \le +1.98) = (.0080 + 0.4761) = 0.4841$
10. $P(+.25 \le z \le +.75) = (0.2734 - 0.0987) = 0.1747$
11. $0.49865 + 0.49865 = 0.9973$
12. $0.2190 + 0.2190 = 0.4380$
13. $0.4713 - 0.3438 = 0.1275$
14. $0.5 + 0.4788 = 0.9788$
15. $0.5 - 0.4599 = 0.0401$
16. $0.4750 + 0.4750 = .9500$
17. $0.5 + 0.4901 = 0.9901$
18. $0.4495 + 0.4495 = 0.8990$
19. $0.4834 - 0.1808 = 0.3026$

Exercise 15.5, pp. 563-564.

Using $Z = \frac{x - \mu x}{\sigma x}$, with $\mu_x = 40.0$ and $\sigma_x = 9.0$

(a) $Z = (50 - 40)/9 = +1.11$

$P(x \geq 50) = .5 - .3665 = 0.1335$

(b) $Z = (35 - 40)/9 = -.0.56$

$P(35 \leq x \leq 50) = .3665 + .2123 = 0.5758$

$Z = (50 - 40) = +1.11$

(c) $Z = (55 - 40)/9 = +1.67$

$P(x \leq 55) = 0.5 + 0.4525 = 0.9525$

(d) $Z = (30 - 40)/9 = -1.11$

$P(x \geq 30) = 0.5 + 0.3665 = 0.8665$

(e) $Z = (45 - 40)/9 = +0.56$ $Z = (55 - 40)/9 = +1.67$

$P(45 \leq x \leq 55) = 0.4525 - 0.2123 = 0.2402$

(f) $Z = (43.5 - 40)/9 = +0.39$

$Z = (44.5 - 40)/9 = +0.50$

$P(x = 44) = 0.1915 - 0.1517 = 0.0398$

2. With $\mu_x = 65.00$ and $\sigma_x = 4.00$

(a) $Z = (60 - 65)/4 = -1.25$

$Z = (70 - 65)/4 = +1.25$

$P(60 \leq x \leq 70) = 0.3944 + 0.3944 = 0.7888$

(b) $Z = (74 - 65)/4 = +2.25$

$P(x > 74) = 0.5 - 0.4878 = 0.0122$

(c) $Z = (65 - 65)/4 = 0$

$P(x \leq 65) = 0.5000$

(d) $Z = (68 - 65)/4 = +0.75$

$Z = (72 - 65)/4 = +1.75$

$P(68 \le x \le 72) = 0.4599 - 0.2734 = 0.1865$

(e) $Z = (60 - 65)/4 = -1.25$

$P(x > 60) = 0.5 + 0.3944 = 0.8944$

3. With $\mu_x = 450.0$ and $\sigma_x = 5.0$

(a) $Z = (448 - 450)/5 = -0.4$

$Z = (453 - 450)/5 = +0.6$

$P(448 \le x \le 453) = 0.1554 + 0.2257 = 0.3811$

(b) $P = (0.90/2) = 0.45$

$Z = \pm 1.645$

$1.645 = (x - 450)/5$

$x - 450 = 8.225$

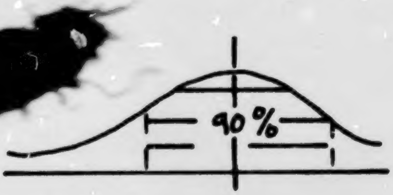
$x = 458.225$

$-1.645 = (x - 450)/5$

$x - 450 = -8.225$

$x = 441.775$

$P(441.775 \le x \le 458.225) = .90$



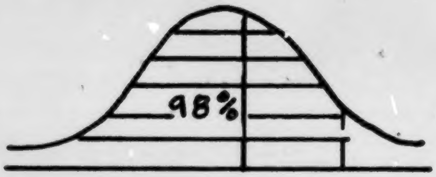
(c) $P = .98 - .5 = .48$

$Z = +2.05$

$+2.05 = (x - 450)/5$

$x - 450 = 10.25$

$x = 460.25$



4. $Z = (49.5 - 45)/5 = 0.9$

$Z = (50.5 = 45)/5 = 1.1$

$P(x = 50) = P(49.5 \leq x \leq 50.5) = 0.3643 - 0.3159 = 0.0484$

5. low speed: $Z = (57.96 - 60)/.5 = -0.408$

high speed: $Z = (2.988 - 3)/.04 = -0.30$

Neither requires service; both are faster than average.

6. In order to make a meaningful comparison, you would need to know the mean and standard deviation for the regular class's performance.

7. (a) $Z = (100 - 73.8)/12.6 = +2.08$

$Z = (90 - 73.8)/12.6 = +1.29$

$P(x \geq 90) = 0.4812 - 0.4015 = 0.0797$

(b) $Z = (89.9 - 73.8)/12.6 = +1.28$

$Z = (80 - 73.8)/12.6 = +0.49$

$P(89.9 \leq x \leq 80) = 0.3997 - 0.1879 = 0.2118$

(c) $Z = (79.9 - 73.8)/12.6 = +0.48$

$Z = (70 - 73.8)/12.6 = -0.30$

$P(79.9 \leq x \leq 70) = 0.1844 - 0.1179 = 0.3023$

(d) $Z = (69.9 - 73.8)/12.6 = -0.31$

$Z = (60 - 73.8)/12.6 = -1.09$

$P(69.9 \leq x \leq 60) = 0.3621 - 0.1271 = 0.2350$

(e) $Z = (59.9 - 73.8)/12.6 = -1.10$

$P(x \leq 59.9) = 0.5 - 0.3643 = 0.1357$

Multiplying each area by n of 50,

A = 3.99

B = 10.59

C = 15.12

D = 11.75

F = 6.79

Rounded:
A = 4
B = 11
C = 15
D = 12
F = 7

8. With $\mu_x = 8.06$ and $\sigma_x = 0.03$

(a) $Z = (7.99 - 8.06)/0.03 = -2.33$

$P(x < 8) = 0.5 - 0.4901 = 0.0099$

$(100)P(x < 8) = 0.0099 \times 100 = 0.99 = 1$

(b) $(8.105 - 8.06)/0.03 - +1.50$

$P(x > 8.1) = 0.5 - 0.4332 = 0.0668$

$(100)P(x > 8.1) = 0.0668 \times 100 = 6.68 = 7$

(c) $Z = (7.92 - 8.06)/0.03 = -4.67$

$Z = (8.08 - 8.06)/0.03 = +0.67$

$1 - P(7.92 \le x \le 8.08) = 1 - (0.5 + 0.2486) = 0.2514$

$(100)P(7.92 \le x \le 8.08) = 0.2514 \times 100 = 25.14 = 25$

9. $Z = (98 - 110)/10 = -1.2$

$Z = (120 - 110)/10 = +1.00$

$P(98 \le x \le 120) = 0.3849 + 0.3413 = 0.7262$

$n(0.7262) = 2$

$n = 2.75$, thus 3 people

10. $\mu_A = 10$ $\sigma_B = 15$

$\mu_A = 2$ $\sigma_B = 6$

	Worker A	Worker B
$\mu_x \pm 1\sigma_x$	8-12 units	9-21 units
$\mu_x \pm 2\sigma_x$	6-14 units	3-27 units
$\mu_x \pm 3\sigma_x$	4-16 units	0-33 units

Which worker is preferred depends on many considerations, some of which are:

1. Are the distributions normally distributed? Perhaps they are skewed.
2. What is the complexity of the work?
3. What is the cost per unit? Perhaps we need to be failure avoidance oriented rather than success oriented.

4. In statistics, the ratio of σ_x and μ_x are compared. The ratio, called "coefficient of variation," equals $(\sigma_x/\mu_x) \times 100$.

$CV_A = (2/10)100 = 20\%$ $CV_B = (6/15)100 = 40\%$

Worker A is less risky than worker B.

REORDER NUMBER 13-2386